Research Statement

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Introduction

My primary area of research is the study of monodromy, a measure of the variation of cohomology in a family of curves. Stemming from ideas in topology and algebraic geometry, monodromy has far reaching connections to number theory, and arithmetic groups. My other research interests involve the application of algebraic geometry to machine learning and the development of practical active learning systems.

This report is broken into three sections:

1. Big mod ℓ monodromy for families of G covers.
2. The Ceresa cycle for Picard curves.
3. Active learning and the pairwise comparison embedding problem.

Along the way I will discuss connections to topics in the theory of groups of Lie type, tropical geometry, Galois representations and combinatorial rigidity theory.

Big Mod ℓ Monodromy for Families of G Covers

Background. Let $X \to S$ be a proper family of curves of genus ≥ 1 with S connected. Given a point $s \in S$, the fundamental group $\pi_1(S, s)$ acts linearly on the cohomology, $H^1(X_s, \mathbb{Z})$, of the fiber X_s and preserves the symplectic intersection form. There is an associated monodromy representation

$$\rho : \pi_1(S, s) \to \mathrm{Sp}(H^1(X_s, \mathbb{Z}/\ell))$$

whose image is the mod ℓ *monodromy group* or the *monodromy* of the family. Intuitively, we are "moving" a cohomology class along a loop based at s. Monodromy representations arise in many different areas such as in the study of the mapping class group and Cohen-Lenstra heuristics for function fields. In general we are interested in situations where the monodromy group is "big" i.e. "as large as it could be given any constraints."

In my thesis I study moduli spaces of connected curves that are Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ with fixed Galois group G. Fix a center-free group G, a choice of conjugacy classes $\mathcal{C} = (\mathcal{C}_1, \cdots, \mathcal{C}_k)$ generating G, and a tuple of integers $m = (m_1, \cdots, m_k)$. A curve has type $(G, \mathcal{C}, m)$ if it is a connected G cover of $\mathbb{P}^1_{\mathbb{C}}$, branched at a set of points $E = \cup_i E_i \subset \mathbb{C}$, where the inertia group of a point in E_i is generated by an element of $\mathcal{C}_i$ and $|E_i| = m_i$. Moduli spaces of curves of type $(G, \mathcal{C}, m)$, known as *Hurwitz spaces*, are denoted by $\mathrm{CHur}^{\mathcal{C}}_{G,m}$. Since G was chosen to be center free, $\mathrm{CHur}^{\mathcal{C}}_{G,m}$ is a fine moduli space and there is a universal curve whose fiber over a closed point x is a corresponding complex curve. We are interested in the mod ℓ monodromy of the universal curve over $\mathrm{CHur}^{\mathcal{C}}_{G,m}$. For this family, the image of monodromy cannot be the full symplectic group and instead must lie in the subgroup of the symplectic group commuting with the action of G, which we will refer to as the mod ℓ *symplectic centralizer*.

Definition We say a triple $(G, \mathcal{C}, m)$ has *big monodromy* mod ℓ if, for every connected component of the Hurwitz space $\mathrm{CHur}_{G,m}^{\mathcal{C}}$, the image of the mod ℓ monodromy representation corresponding to the universal curve contains the commutator subgroup of the mod ℓ symplectic centralizer.

In general the mod ℓ symplectic centralizer decomposes as a product of orthogonal, unitary, symplectic, and general linear groups. So for example, if one of these components is unitary, big monodromy implies that the monodromy group contains the special unitary subgroup in that factor.

Example. A few specific cases have been worked out when $G = \mathbb{Z}/d$. Let $\{i\}$ be the conjugacy class of the element $i \in \mathbb{Z}/d$. Curves of type $(\mathbb{Z}/2, (\{1\}), m)$ are hyperelliptic curves of genus $2m + 2$ and the associated Hurwitz space is the moduli space of these hyperelliptic curves. In this case, the symplectic centralizer is the full symplectic group since the hyperelliptic involution acts on homology as multiplication by -1. For $\ell \neq 2$, A'Campo, Achter and Pries, Hall and J.K. Yu ([1], [2], [12], [23]) have independently shown that $(\mathbb{Z}/2, (\{1\}), m)$ has big monodromy mod ℓ. In the other direction Brendle and Margalit [4] have studied congruence subgroups of $\pi_1(\mathrm{CHur}_{G,m}^{\mathcal{C}})$ arising from the mod 2 and mod 4 monodromy representations. When $G = \mathbb{Z}/3$, Achter and Pries show that both $(G, (\{1\}), m)$ and $(G, (\{1\}, \{2\}), m)$ have big monodromy mod ℓ for $\ell \geq 5$ if m is chosen so that the genus of these curves is at least 3.

Building on work of Deligne and Mostow, McMullen [15] studies the monodromy with $\mathbb{Z}$ coefficients and shows that for curves of type $(\mathbb{Z}/4, (\{1\}), m = 18)$ the monodromy group gives an example of a *thin* group, i.e. it has infinite index in its Zariski closure. However when $m \geq 2d$ the resulting monodromy group is an arithmetic lattice by recent results of Venkataramana ([20], [21]). This implies that the mod ℓ monodromy group will be the full symplectic centralizer for sufficiently large ℓ.

Previous Work: Little work has been done to generalize these monodromy results to families of curves that are covers of $\mathbb{P}^1$ for non-cyclic groups. My theorem, inspired by former work in the cyclic case, attempts to remedy this. I restrict to groups G with trivial Schur multiplier. This implies that $\mathrm{CHur}_{G,m}^{\mathcal{C}}$ will be connected for any choice of conjugacy classes $\mathcal{C}$ [22].

Theorem 1 (Jain, L.). *Let G be a center-free group with trivial Schur multiplier. For $\ell \nmid 2|G|$, $m = (m_1, \cdots, m_k)$ with each m_i sufficiently large, $(G, \mathcal{C}, m)$ has big monodromy for any choice of $\mathcal{C}$.*

There are two key ingredients going into the proof. Firstly, the monodromy group naturally acts on a set of particular G-invariant subspaces U of $H^1(C, \mathbb{Z}/\ell\mathbb{Z})$. The orbits of this action can be interpreted as connected components of Hurwitz schemes of $G \ltimes U$ covers of $\mathbb{P}^1_{\mathbb{C}}$. Using results due to Ellenberg, Venkatesh, and Westerland (EVW) [22] (based on work of Biggers and Fried), there are group theoretic criteria that can be used to compute the number of such components. Secondly, we can apply results from the study of transitive subgroups of classical groups of Lie type over finite fields. Results of Cameron-Kantor and others ([5]) show that subgroups of a classical group Γ with the same number of orbits on subspaces as Γ must contain contain the commutator $[\Gamma, \Gamma]$.

This strategy differs greatly from those used by Hall and McMullen who both construct explicit transvections in the monodromy group. This is best explained through analogy with permutation groups. One way to show that a transitive group is a "large" subgroup of S_n is to explicitly show that it contains a transposition and n-cycle and hence is all of S_n. Alternatively we can study the action; if the group acts at least 6 transitively on a set, it must be A_n or S_n [6]. The first technique is analogous to the kind of group theory input used for monodromy results in the past, and the second is closer to the tools I use in the proof of Theorem 1.

Future Work: There are two future directions that I hope to pursue.

1. In the statement of the theorem above, it is assumed that G is center-free and has trivial Schur multiplier.

   Problem 1. *Remove the restrictions on G in Theorem 1.*

   This will require analogs of the results in EVW to Hurwitz stacks and a more detailed group theoretic analysis.
2. The number of branching points needed in the statement of the theorem is dependent on the choice of prime. In the case of cyclic covers, Venkataramana's results show that for a fixed number of branch points, the mod ℓ monodromy is big for ℓ sufficiently large; but it is not obvious how the bound on ℓ depends on m. This leads to the following question:

   Problem 2. *Make the dependence between the number of branching points needed for Theorem 1 and ℓ precise.*

   This will require an analysis of braid group actions on Nielsen Classes that give rise to the formulas for the number of connected components of Hurwitz schemes.

THE CERESA CYCLE FOR PICARD CURVES

Background Let C be a smooth proper curve over a characteristic zero field K and let c be a point on $C(K)$. The Abel-Jacobi map sending a point $x \in C$ to the divisor class $x - c$ is an embedding of $C \to \mathrm{Jac}(C)$. The image of this map is an algebraic cycle $[C]$ and there is an associated cycle $[C^-] = i_*[C]$ where i is the involution of $\mathrm{Jac}(C)$ mapping each point to its inverse. The Ceresa cycle is defined as $[C] - [C^-] \in \mathrm{Ch}_1(\mathrm{Jac}(C))$, where $\mathrm{Ch}_1(\mathrm{Jac}(C))$ is the group of dimension 1 algebraic cycles on the Jacobian modulo algebraic equivalence.

For a hyperelliptic curve, the Ceresa cycle is algebraically equivalent to zero since the action of the hyperelliptic involution agrees with negation on the Jacobian. An early result of Ceresa [8] showed that for a generic curve of genus greater than three the Ceresa cycle is not algebraically equivalent to zero, providing one of the first examples of an algebraic cycle proven to be homologically trivial but not algebraically trivial.

Future Work: As far as I know there are no explicit examples for when the Ceresa cycle is torsion. In the smallest non trivial case of genus three curves, there is a two dimensional family of curves known as Picard curves arising as $\mathbb{Z}/3$ covers of $\mathbb{P}^1$ and given by an equation $y^3 = f(x)$ with $f(x)$ a degree four polynomial. In this case there is the following conjecture [16]:

Conjecture 1. *For C a general Picard curve, $3([C] - [C^-])$ is algebraically equivalent to zero.*

I propose two possible avenues of study that could provide evidence towards the conjecture.

1. When K is a number field the Ceresa cycle is related to a cycle class in $H^1(G_K, \Lambda^3 H_1(C \otimes \bar{K}, \mathbb{Z}_\ell(1)))$ where G_K is the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. Hain and Matsumoto show that the nonvanishing of this class is governed by the action of G_K on the pro-ℓ completion

of the fundamental group modulo the second term in its lower central series [11]. This action arises from a map $\rho : G_K \to \mathrm{Aut}(\pi_1^{(\ell)}(C \otimes \bar{K}, c))$ which is an arithmetic analogue of the monodromy group mentioned in the previous project. If we let $G_K^{\mathrm{tor}} = \ker \rho$ (the analog of the Torelli subgroup for the action of the Galois group), then by Hain-Matsumoto's ℓ-adic Harris-Pulte theorem, the Ceresa cycle being 3 torsion would imply the kernel of the action of G_K^{tor} on $\pi_1^{(\ell)}(C \otimes \bar{K}, c)/L^3\pi_1^{(\ell)}(C \otimes \bar{K}, c)$ having index 6. I propose to show this by studying the Galois action directly. This would provide strong evidence for the conjecture.

2. We can also consider the case when K is algebraically closed and complete with respect to a non-trivial non-Archimedean valuation and apply tools from *tropical geometry*. Ilia Zharkov [24] has provided a tropical analogue of the Ceresa cycle for metric graphs and their associated Jacobians and proved an appropriate analogue of Ceresa's theorem in this case. He introduces a new tropical tool, the determinantal form, that can be used to combinatorially check algebraic inequivalence of cycles on a metric graph. I propose applying these tropical analogues to the conjecture.

   However, it is important that the Berkovich skeleton of the Picard curve is actually a genus three metric graph, or in other words, that the curve is totally degenerate. Even if this condition is met, the tropicalization of the standard planar embedding of a Picard curve carries little information about cycle classes in the Jacobian. So, an embedding of the curve has to be found that gives a faithful representation of the Berkovich skeleton of its analytification. Hence a first problem to solve would be:

   Problem 3. *Given a totally degenerate Picard curve over K, find an embedding that gives a faithful tropicalization.*

   This problem is difficult for a general curve. Chan and Sturmfels [9] study the case of elliptic curves and provide an explicit algorithm for finding a planar embedding that has a faithful tropicalization in a symmetric honeycomb form. In recent work, Cueto, Markwig and Morrison use *tropical Modifications* to analyze the case of genus two curves. The case of Picard curves would be the next natural case to work out.

Active Learning and the pairwise comparison embedding problem

I am extremely interested in machine learning and its connections to algebra. A particular area of interest is *active learning*: the development and implementation of algorithms for adaptive data collection.

Previous Work: NEXT system for active learning Active learning algorithms automatically adjust what data to collect next based on previously collected data. This is especially useful when data collection can be prohibitively expensive, for example when extensive labeling of data by humans is required. They can be used in tandem with many standard machine learning tasks such as classification, clustering or embedding. However actually implementing systems to do active learning is an extremely difficult engineering problem and the gap between theory and practice is large. Such software requires a cloud-based framework to deliver actively chosen queries, a database to store answers, and a computational backend to update a model and generate queries. For many researchers seeking to use active learning, this is an insurmountable obstacle. To address these issues, I worked with Kevin Jamieson and Robert Nowak to develop NEXT: a unique platform and cloud service for large-scale, reproducible active learning research. NEXT enables machine learning researchers to easily deploy and test new active learning algorithms and applied researchers to employ active learning methods for real-world applications. Our proposal for NEXT was recently awarded a spotlight at the 2015 Neural Information Processing Systems (NIPS) conference [10], an influential machine learning conference.

Future Work: One of the main applications implemented in NEXT is related to *multidimensional scaling (MDS)*. In MDS, there are a set of items embedded in $\mathbb{R}^d$ with their precise locations unknown. At each step of an algorithm, an oracle (in practice a human taking a survey) is queried to reveal some information about the distances between the points. These queries are made *actively* rather than at random.

Problem 4. *How many distances need to be actively observed to recover the location of the points up to an affine orthogonal transformation?*

A set of points in Euclidean space is determined by all pairwise distances between the points up to an affine transformation, so certainly knowing all distances would suffice. However if questions are asked actively, fewer distances should be needed. This problem is inherently linked to *low rank matrix completion*. The distance matrix of the set of points (where the i, j-th entry is the distance from point i to point j) has rank at most $d+2$. Using the standard bounds due to Candes and Recht [7], one would expect that $O(dn \log n)$ *random* distances should suffice. Applications of algebraic geometry to matrix completion have been studied by Pimentel-Alarcòn, Boston and Nowak [17], and Király, Rosen, and Theran [19] among others.

In real life applications people can rarely pin down an exact distance between items. However, humans are excellent at providing relative judgements of the form“item i is closer to item j than item k”. So the hope is to *closely recover* a set of points, i.e. recover the set of points up to an affine orthogonal transformation within a given error tolerance. This leads to the following relaxation of the previous problem:

Problem 5. *How many active queries of the form “which item is closer to item i, item j or item k”, closely recover an embedding of the items?*

Note, it is not immediately clear what sets of points can even be closely recovered. Though there are many proposed algorithms (e.g. [13]) few papers have discussed theoretical bounds on how many queries would be needed for close recovery. Notable exceptions to this are the paper of Jamieson and Nowak where they use hyperplane arrangements intersecting a Veronese subvariety to demonstrate that at least $\Omega(dn \log(n))$ actively chosen queries are required. Results of von Luxburg and Kleindessner [14] and Arias-Castro [3] show consistency of the resulting embedding as the number of points gets large.

Both problems are linked to the *combinatorial rigidity of frameworks*. A framework is a graph, $G = (V, E)$ along with a edge lengths $\ell_{i,j}$ for each choice of edge $i, j \in E$. A graph is considered *rigid* if there are only a finite set of embedding of the graph in Euclidean space up to rigid motions. Away from a measure zero choice of edge lengths, $\ell_{i,j}$, rigidity is determined by G alone and is a completely combinatorial criterion. The main problem in rigidity theory is to characterize exactly what graphs are rigid. There are also deep connections between rigidity and matroid theory that have been applied to matrix completion problems [18]. I hope to use combinatorial rigidity theory, and techniques from real algebraic geometry to provide upper bounds to the number of active queries needed in each problem.

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