Limits, Math 221 Do as many as you can!

1. Evaluate the limit $\lim_{x\to 4} x^4 - 4x + 1$.

   Answer:

$$\begin{aligned}\lim_{x\to 4} x^4 - 4x + 1 &= \left(\lim_{x\to 4} x\right)^4 - 4\lim_{x\to 4} x + \lim_{x\to 4} 1\\ &= 4^4 - 4(4) + 1 = 241\end{aligned}$$

2. Evaluate the limit $\lim_{x\to 3} \dfrac{3x^2+1}{\sqrt{x^3+9}}$.

   Answer:

$$\begin{aligned}\lim_{x\to 3} \frac{3x^2+1}{\sqrt{x^3+9}} &= \frac{3(\lim_{x\to 3} x)^2 + \lim_{x\to 3} 1}{((\lim_{x\to 3} x)^3 + \lim_{x\to 3} 9)^{1/2}}\\ &= \frac{3(3)^2+1}{(3^3+9)^{1/2}} = \frac{14}{3}\end{aligned}$$

3. Evaluate the limit $\lim_{x\to 1} \dfrac{x^2-2x+1}{x^2-3x+2}$.

   Answer:

$$\begin{aligned}\lim_{x\to 1} \frac{x^2-2x+1}{x^2-3x+2} &= \lim_{x\to 1} \frac{(x-1)^2}{(x-1)(x-2)}\\ &= \lim_{x\to 1} \frac{x-1}{x-2}\\ &= \frac{1-1}{1-2} = 0\end{aligned}$$

4. Evaluate the limit $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$.

Answer:

$$\begin{aligned}\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}\\ &= \lim_{x\to 4}\sqrt{x}+2\\ &= \sqrt{4}+2=4\end{aligned}$$

5. Evaluate the limit $\lim_{x\to 1}\dfrac{2x-1}{8x^3-1}$

Answer:

$$\begin{aligned}\lim_{x\to 1}\frac{2x-1}{8x^3-1} &= \lim_{x\to 1}\frac{2x-1}{(2x-1)(4x^2+2x+1)}\\ &= \lim_{x\to 1}\frac{1}{4x^2+2x+1}\\ &= \frac{1}{4+2+1}=\frac{1}{7}\end{aligned}$$

6. Evaluate the limit $\lim_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

Answer:

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} &= \lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2}\\ &= \lim_{x\to 2}\frac{2-x}{(2x)(x-2)}\\ &= \lim_{x\to 2}\frac{-1}{2x}\\ &= -\frac{1}{4}\end{aligned}$$

7. Evaluate the limit $\lim_{x\to 3} \dfrac{\sqrt{x}-3}{x-9}$.

8. Evaluate the limit $\lim_{x\to 3} \dfrac{\sqrt{x+6}-3}{x^2-9}$ or show that it does not exist.

Answer:

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-9} &= \lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-9} \cdot \frac{\sqrt{x+6}+3}{\sqrt{x+6}+3} \\
&= \lim_{x\to 3} \frac{(x+6)-9}{(x^2-9)(\sqrt{x+6}+3)} \\
&= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)(\sqrt{x+6}+3)} \\
&= \lim_{x\to 3} \frac{1}{(x+3)(\sqrt{x+6}+3)} \\
&= \frac{1}{(3+3)(\sqrt{9}-3)} = \frac{1}{36}
\end{aligned}$$

9. Evaluate the limit $\lim_{x\to \infty} \dfrac{x^3+2x+1}{3x^3-x+4}$ or show that it does not exist.

Answer:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+2x+1}{3x^3-x+4} &= \lim_{x\to\infty} \frac{x^3\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)}{x^3\left(3-\frac{1}{x^2}+\frac{4}{x^3}\right)} \\
&= \lim_{x\to\infty} \frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{3-\frac{1}{x^2}+\frac{4}{x^3}} \\
&= \frac{1}{3}
\end{aligned}$$

10. Evaluate the limit $\lim_{x\to-\infty} \frac{x^{17}-7x^{2013}}{3x^2+x^{2014}}$ or show that it does not exist.

Answer:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^{17}-7x^{2013}}{3x^2+x^{2014}} &= \lim_{x\to-\infty} \frac{x^{2013}\left(\dfrac{1}{x^{1996}}-7\right)}{x^{2014}\left(\dfrac{3}{x^{2012}}+1\right)} \\
&= \lim_{x\to-\infty} \frac{1}{x} \frac{\dfrac{1}{x^{1996}}-7}{\dfrac{3}{x^{2012}}+1} \\
&= 0 \cdot \frac{-7}{1} = 0
\end{aligned}$$

11. Evaluate the limit $\lim_{x\to\infty} \frac{\sqrt{x^2+3x}}{2x}$ or show that it does not exist.

Answer:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+3x}}{2x} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x}\right)}}{2x} \\
&= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{3}{x}}}{2x} \\
&= \lim_{x\to\infty} \frac{|x|}{x} \cdot \frac{1}{2}\sqrt{1+\frac{3}{x}} \\
&= \frac{1}{2}
\end{aligned}$$

The last step uses the fact that $|x| = x$ as $x \to \infty$, so $\lim_{x\to\infty} \frac{|x|}{x} = 1$.

12. Evaluate the limit $\lim_{x\to-\infty} \frac{x-3}{\sqrt{x^2-9}}$ or show that it does not exist.

Answer:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x-3}{\sqrt{x^2-9}} &= \lim_{x\to-\infty} \frac{x\left(1-\dfrac{3}{x}\right)}{\sqrt{x^2\left(1-\dfrac{9}{x^2}\right)}} \\
&= \lim_{x\to-\infty} \frac{x\left(1-\dfrac{3}{x}\right)}{|x|\sqrt{1-\dfrac{9}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{x}{|x|}\frac{1-\dfrac{3}{x}}{\sqrt{1-\dfrac{9}{x^2}}} \\
&= -1
\end{aligned}$$

The last step uses the fact that $|x| = -x$ as $x \to -\infty$, so $\lim_{x\to-\infty} \frac{x}{|x|} = -1$.

13. Evaluate the limit $\lim_{x\to\infty} \sqrt{x^2+x} - x$

Answer:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x} - x &= \lim_{x\to\infty} \sqrt{x^2+x} - x \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x} \\
&= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}}+x} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \\
&= \frac{1}{2}
\end{aligned}$$

14. Here are your tasks. Some might be impossible. Find functions $f(x)$ and $g(x)$ so that

(a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ but $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$.

(b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ but $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 2$.

(c) $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ but $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$.

(d) $\lim_{x\to 0} f(x) = 0, \lim_{x\to 0} g(x) = +\infty$ but $\displaystyle\lim_{x\to 0} f(x)g(x) = 3$.

(e) $\lim_{x\to 0} f(x) = 0, \lim_{x\to 0} g(x) = +\infty$ but $\displaystyle\lim_{x\to 0} f(x)g(x) = -3$.

(f) $\lim_{x\to 0} f(x) = 0, \lim_{x\to 0} g(x) = +\infty$ but $\displaystyle\lim_{x\to 0} f(x)g(x) = 3$.

(g) $\lim_{x\to 0} f(x) = \infty, \lim_{x\to 0} g(x) = -\infty$ but $\displaystyle\lim_{x\to 0} f(x) + g(x) = 4$.