Trig Limits, Math 221 Do as many as you can!

1. Recall that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \). Use this limit along with the other “basic limits” to find the following:

(a) \( \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \). [Hint: Multiply top and bottom by \( 1 + \cos(x) \).]

(b) \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} \).

(c) \( \lim_{x \to 0} \frac{\tan(x)}{x} \).

Answer:

(a)

\[
\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} \\
= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x^2 (1 + \cos(x))} \\
= \lim_{x \to 0} \frac{\sin^2(x)}{x^2 (1 + \cos(x))} \\
= \lim_{x \to 0} \left( \frac{\sin(x)}{x} \right)^2 \left( \frac{1}{1 + \cos(x)} \right) \\
= 1 \cdot \frac{1}{1 + 1} = \frac{1}{2} 
\]

(b) Using (a),

\[
\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} x \cdot \frac{1 - \cos(x)}{x^2} \\
= 0 \cdot \frac{1}{2} = 0 
\]

(c)

\[
\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{x} \\
= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} \\
= 1 \cdot \frac{1}{1} = 1 
\]
2. Evaluate the limit \( \lim_{x \to \frac{\pi}{2}} \tan(x) - \sec(x) \) or show that it does not exist. Answer:

\[
\lim_{x \to \frac{\pi}{2}} \tan(x) - \sec(x) = \lim_{x \to \frac{\pi}{2}} \frac{\sin(x) - 1}{\cos(x)} \cdot \frac{\sin(x) + 1}{\sin(x) + 1}
\]

\[
= \lim_{x \to \frac{\pi}{2}} \frac{\sin^2(x) - 1}{\cos(x)(\sin(x) + 1)}
\]

\[
= \lim_{x \to \frac{\pi}{2}} \frac{-\cos^2(x)}{\cos(x)(\sin(x) + 1)}
\]

\[
= \lim_{x \to \frac{\pi}{2}} \frac{-\cos(x)}{\sin(x) + 1}
\]

\[
= \frac{0}{2} = 0
\]

3. Evaluate the limit \( \lim_{x \to 2014\pi} \frac{\sin(x + 2013\pi)}{\sin(x)} \) or show that it does not exist. Answer: None.

\[
\lim_{x \to 2014\pi} \frac{\sin(x + 2013\pi)}{\sin(x)} = \lim_{x \to 2014\pi} \frac{\sin(x)\cos(2013\pi) + \sin(2013\pi)\cos(x)}{\sin(x)}
\]

\[
= \lim_{x \to 2014\pi} -\frac{\sin(x)}{\sin(x)}
\]

\[
= \lim_{x \to 2014\pi} -1
\]

\[
= -1.
\]

4. Evaluate the limit \( \lim_{x \to 0} \frac{\sin(x)}{1 - \cos(x)} \) or show that it does not exist. Answer:

\[
\lim_{x \to 0} \frac{\sin(x)}{1 - \cos(x)} = \lim_{x \to 0} \frac{\sin(x)}{1 - \cos(x)} \cdot \frac{1 + \cos(x)}{1 + \cos(x)}
\]

\[
= \lim_{x \to 0} \frac{\sin(x)(1 + \cos(x))}{1 - \cos^2(x)}
\]

\[
= \lim_{x \to 0} \frac{\sin(x)(1 + \cos(x))}{\sin^2(x)}
\]

\[
= \lim_{x \to 0} \frac{1 + \cos(x)}{\sin(x)}
\]

As \( x \to 0 \), \( 1 + \cos(x) \to 2 \) and \( \sin(x) \to 0 \). As \( x \) approaches 0 from the positive side, \( \sin(x) > 0 \). As \( x \) approaches 0 from the negative side, \( \sin(x) < 0 \). This shows that

\[
\lim_{x \to 0^+} \frac{1 + \cos(x)}{\sin(x)} = \infty
\]

\[
\lim_{x \to 0^-} \frac{1 + \cos(x)}{\sin(x)} = -\infty,
\]

so the limit does not exist.
5. Evaluate the limit \( \lim_{x \to 0} \frac{1 - \cos(3x)}{2x^2} \) or show that it does not exist. Answer:

\[
\lim_{x \to 0} \frac{1 - \cos(3x)}{2x^2} = \lim_{x \to 0} \frac{1 - \cos(3x)}{(3x)^2} \cdot \frac{(3x)^2}{2x^2} \\
= \frac{9}{2} \lim_{x \to 0} \frac{1 - \cos(3x)}{(3x)^2} \\
= \frac{9}{2} \cdot \frac{1}{2} = \frac{9}{4}.
\]

6. Evaluate the limit \( \lim_{x \to 0} \frac{3x}{\sin(2x)} \) or show that it does not exist. Answer:

\[
\lim_{x \to 0} \frac{3x}{\sin(2x)} = \lim_{x \to 0} \frac{3 \cdot 2x}{2 \sin(2x)} \\
= \frac{3}{2} \lim_{x \to 0} \left( \frac{\sin(2x)}{2x} \right)^{-1} \\
= \frac{3}{2}.
\]

7. Evaluate the limit \( \lim_{x \to 0} \frac{\sin(\sin(x))}{x} \). Answer:

\[
\lim_{x \to 0} \frac{\sin(\sin(x))}{x} = \lim_{x \to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \frac{\sin(x)}{x} \\
= \lim_{x \to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \lim_{x \to 0} \frac{\sin(x)}{x} \\
= 1 \cdot 1 = 1.
\]

8. Evaluate the limit \( \lim_{x \to 0} \frac{1 - \cos(\pi x)}{x^2} \). Answer:

\[
\lim_{x \to 0} \frac{1 - \cos(\pi x)}{x^2} = \lim_{x \to 0} \frac{1 - \cos(\pi x)}{\pi^2 x^2} \cdot \pi^2 \\
= \pi^2 \lim_{x \to 0} \frac{1 - \cos(\pi x)}{(\pi x)^2} \\
= \pi^2 \cdot \frac{1}{2} = \frac{\pi^2}{2}.
\]

9. Evaluate the limit \( \lim_{x \to 0} \frac{\sin(3x^2)}{x} \) Answer:

\[
\lim_{x \to 0} \frac{\sin(3x^2)}{x} = \lim_{x \to 0} \frac{\sin(3x^2)}{x} \cdot \frac{3x}{3x} \\
= \lim_{x \to 0} \frac{\sin(3x^2)}{3x^2} \cdot 3x \\
= 0
\]
10. Evaluate the limit \( \lim_{x \to 0} \frac{\sin(x)}{x + \tan(x)} \) Answer:

\[
\lim_{x \to 0} \frac{\sin(x)}{x + \tan(x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{1 + \frac{\sin(x)}{x \cos(x)}} = \frac{1}{1 + 1} = \frac{1}{2}
\]

11. Evaluate the limit \( \lim_{x \to 0} \frac{1 - \cos(x)}{x \sin(x)} \) Answer:

\[
\lim_{x \to 0} \frac{1 - \cos(x)}{x \sin(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \frac{x \sin(x)}{x^2} = \frac{1}{2} \cdot 1 = \frac{1}{2}
\]

12. Evaluate the limit \( \lim_{x \to 0} \frac{\sin(x) \sin(2x)}{\sin(3x) \sin(4x)} \). Answer:

\[
\lim_{x \to 0} \frac{\sin(x) \sin(2x)}{\sin(3x) \sin(4x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(2x)}{2x} \cdot \frac{3x}{4x} \frac{1}{6} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{6} = \frac{1}{6}.
\]

13. Evaluate the limit \( \lim_{x \to 0} \frac{1 - \cos(5x)}{\sin^2(3x)} \). Answer:

\[
\lim_{x \to 0} \frac{1 - \cos(5x)}{\sin^2(3x)} = \lim_{x \to 0} \frac{1 - \cos(5x)}{x^2} \cdot \frac{(3x)^2}{\sin^2(3x)} \cdot \frac{(5x)^2}{(3x)^2} = \frac{25}{9} \cdot \frac{1}{2} \cdot 1 = \frac{25}{18}.
\]

14. Use the Sandwich Theorem to evaluate the limit \( \lim_{x \to 0} x \cdot \sin \left( \frac{1}{x} \right) \). Answer: Since \(-1 \leq \sin \left( \frac{1}{x} \right) \leq 1\) for all \(x\), it follows that \(-x \leq x \sin \left( \frac{1}{x} \right) \leq x\) for all \(x\).
We know that \( \lim_{x \to 0} |x| = 0 \) and \( \lim_{x \to 0} -|x| = 0 \). Therefore, the Sandwich Theorem says that 
\[
\lim_{x \to 0} x \cdot \sin \left( \frac{1}{x} \right) = 0.
\]

15. Use the definition of the derivative to find the derivative of \( f(x) = 3x + 2 \). Answer:

\[
f'(x) = \lim_{h \to 0} \frac{(3(x+h) + 2) - (3x + 2)}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3.
\]

16. Determine where each of the following functions is continuous, and justify your answers:

(a) \( g(x) = \begin{cases} 
(x^2 - 1)/(x+1) & \text{for } x \neq -1 \\
2 & \text{for } x = -1.
\end{cases} \)

(b) \( h(x) = \begin{cases} 
(x^2 - 1)/(x+1) & \text{for } x \neq -1 \\
-2 & \text{for } x = -1.
\end{cases} \)

(c) \( j(x) = \begin{cases} 
x^2 - 2x & \text{for } |x| > 1 \\
x^2 - 2 & \text{for } |x| \leq 1.
\end{cases} \)

(d) \( p(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right) & \text{for } x \neq 0 \\
0 & \text{for } x = 0
\end{cases} \)

(e) \( q(x) = \begin{cases} 
x^2 \sin \left( \frac{1}{x} \right) & \text{for } x \neq 0 \\
0 & \text{for } x = 0
\end{cases} \)

(f) \( k(x) = \lceil x \rceil \), the “ceiling function” (which returns the smallest integer that is \( \geq x \)).

Answer:

(a) \( g(x) \) is continuous at \( x \neq -1 \). At \( x = -1 \), \( f(-1) = 2 \) and

\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x - 1)(x + 1)}{x + 1} = \lim_{x \to -1} x - 1 = -2.
\]

Since \( \lim_{x \to -1} f(x) \neq f(-1) \), the function is discontinuous at \( x = -1 \).
(b) $h(x)$ is continuous at all $x$. Notably, at $x = -1$, $h(-1) = -2$ and

$$\lim_{x \to -1} h(x) = \lim_{x \to -1} \frac{x^2 - 1}{x + 1}$$

$$\quad = \lim_{x \to -1} \frac{(x - 1)(x + 1)}{x + 1}$$

$$\quad = \lim_{x \to -1} x - 1$$

$$\quad = -2.$$

Since $\lim_{x \to -1} h(x) = h(-1)$, the function is continuous at $x = -1$.

(c) $j(x)$ is continuous at $x \neq 1, -1$. At $x = -1$,

$$\lim_{x \nearrow -1} j(x) = \lim_{x \nearrow -1} 3x - 2 = -5$$

$$\lim_{x \searrow -1} j(x) = \lim_{x \searrow -1} x^2 - 2x = 3.$$

Since $\lim_{x \nearrow -1} j(x) \neq \lim_{x \searrow -1} j(x)$, $j$ is discontinuous at $x = -1$.

At $x = 1$,

$$\lim_{x \nearrow 1} j(x) = \lim_{x \nearrow 1} x^2 - 2 = -1$$

$$\lim_{x \searrow 1} j(x) = \lim_{x \searrow 1} 3x - 2 = 1.$$

Since $\lim_{x \nearrow 1} j(x) \neq \lim_{x \searrow 1} j(x)$, $j$ is discontinuous at $x = 1$.

(d) $p(x)$ is continuous at $x \neq 0$. At $x = 0$, $\lim_{x \to 0} p(x) = \lim_{x \to 0} \sin(1/x)$ does not exist.

Thus, the function is discontinuous at $x = 0$.

(e) $q(x)$ is continuous at all $x$. Notably, at $x = 0$, $q(0) = 0$ and

$$\lim_{x \to 0} q(x) = \lim_{x \to 0} x^2 \sin \left(\frac{1}{x}\right) = 0$$

This follows from the Sandwich theorem and the fact that $-x^2 \leq x^2 \sin \left(\frac{1}{x}\right) \leq x^2$.

Therefore $q$ is continuous at $x = 0$.

(f) $k(x)$ is continuous all $x$ not an integer. When $x = n$ where $n$ is an integer,

$$\lim_{x \nearrow n} k(x) = \lim_{x \nearrow n} \lfloor x \rfloor = n + 1$$

$$\lim_{x \searrow n} f(x) = \lim_{x \searrow n} \lfloor x \rfloor = n.$$

Since $\lim_{x \nearrow n} k(x) \neq \lim_{x \searrow n} k(x)$, $k$ is discontinuous at $x = n$.

17. Find a value for $a$ such that the function

$$f(x) = \begin{cases} 
\frac{6x^2 - 54}{x - 3} & \text{for } x \neq 3 \\
 a & \text{for } x = 3
\end{cases}$$
is continuous. Answer: In order for the function $f(x)$ to be continuous at $x = 3$, it needs to be the case that $\lim_{x \to 3} f(x) = f(3)$. In this case,

$$
\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{6x^2 - 54}{x - 3}
$$

$$
= \lim_{x \to 3} \frac{6(x + 3)(x - 3)}{x - 3}
$$

$$
= \lim_{x \to 3} 6(x + 3)
$$

$$
= 6(3 + 3) = 36.
$$

and $f(3) = a$. Therefore, $f(x)$ is continuous when $a = 36$.

18. Find all values of $a$ so that

$$
\begin{align*}
f(x) &= \begin{cases} 
sin(x + a) & \text{for } x < 0 \\
ax^2 & \text{for } x \geq 0
\end{cases}
\end{align*}
$$

is continuous. Answer: In order for the function $f(x)$ to be continuous at $x = 0$, it needs to be the case that $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. In this case,

$$
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} ax^2
$$

$$
= 0
$$

$$
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(x + a)
$$

$$
= \sin(a)
$$

$$
f(0) = a(0)^2 = 0
$$

Therefore $f$ is continuous when $\sin(a) = 0$, so when $a = \pi k$ for any integer $k$.

19. Find all asymptotes (horizontal, vertical, slanted) of the function $f(x) = \frac{3x^2 + x - 1}{x + 3}$. Answer: $\lim_{x \to -3} = \infty$, so $f(x)$ has a vertical asymptote at $x = -3$.

The function $f(x)$ has no horizontal asymptotes.

Using polynomial long division, you can find that

$$
\frac{3x^2 + x - 1}{x + 3} = 3x - 8 + \frac{23}{x + 3}.
$$

Then

$$
\lim_{x \to \infty} \frac{3x^2 + x - 1}{x + 3} - (3x - 8) = \lim_{x \to \infty} \frac{23}{x + 3} = 0.
$$

Therefore $f(x)$ has a slanted asymptote $y = 3x - 8$.

20. Find all asymptotes (horizontal, vertical, slanted) of the function $f(x) = \frac{x^2 - 1}{x^2 + 1}$. Answer:

$$
\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1,
$$

so $f(x)$ has a horizontal asymptotes $y = 1$.

The function $f(x)$ has no vertical or slanted asymptotes.
21. Use the definition of the derivative to find the derivative of $g(x) = x^2$. Answer:

$$g'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$
$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$

22. Use the definition of the derivative to find the derivative of $h(x) = \frac{2}{x}$. Answer:

$$h'(x) = \lim_{h \to 0} \frac{2}{x + h} - \frac{2}{x}$$
$$= \lim_{h \to 0} \frac{2x - 2(x + h)}{(x + h)x}$$
$$= \lim_{h \to 0} \frac{-2h}{h(x + h)x}$$
$$= \lim_{h \to 0} \frac{-2}{(x + h)x}$$
$$= -\frac{2}{x^2}$$

23. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x + 1}$. Answer:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$
$$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$
$$= \lim_{h \to 0} \frac{1}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$
$$= \lim_{h \to 0} \frac{1}{2\sqrt{x+1}}$$