Trig Limits, Math 221 Do as many as you can!

1. Recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Use this limit along with the other "basic limits" to find the following:

   (a) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$. [Hint: Multiply top and bottom by $1+\cos(x)$.]

   (b) $\lim_{x\to 0} \frac{1-\cos(x)}{x}$.

   (c) $\lim_{x\to 0} \frac{\tan(x)}{x}$.

   Answer:

   (a)

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x^2} &= \lim_{x\to 0} \frac{1-\cos(x)}{x^2} \cdot \frac{1+\cos(x)}{1+\cos(x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin^2(x)}{x^2(1+\cos(x))} \\
&= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{1}{1+\cos(x)}\right) \\
&= 1 \cdot \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

   (b) Using (a),

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} &= \lim_{x\to 0} x \cdot \frac{1-\cos(x)}{x^2} \\
&= 0 \cdot \frac{1}{2} = 0
\end{aligned}$$

   (c)

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)}{x} &= \lim_{x\to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} \\
&= 1 \cdot \frac{1}{1} = 1
\end{aligned}$$

2. Evaluate the limit $\lim_{x\to\frac{\pi}{2}} \tan(x) - \sec(x)$ or show that it does not exist. Answer:

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \tan(x) - \sec(x) &= \lim_{x\to\frac{\pi}{2}} \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\sin(x)-1}{\cos(x)} \cdot \frac{\sin(x)+1}{\sin(x)+1} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\sin^2(x)-1}{\cos(x)(\sin(x)+1)} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{-\cos^2(x)}{\cos(x)(\sin(x)+1)} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{-\cos(x)}{\sin(x)+1} \\
&= \frac{0}{2} = 0
\end{aligned}$$

3. Evaluate the limit $\lim_{x\to 2014\pi} \frac{\sin(x+2013\pi)}{\sin(x)}$ or show that it does not exist. Answer: None.

$$\begin{aligned}
\lim_{x\to 2014\pi} \frac{\sin(x+2013\pi)}{\sin(x)} &= \lim_{x\to 2014\pi} \frac{\sin(x)\cos(2013\pi)+\sin(2013\pi)\cos(x)}{\sin(x)} \\
&= \lim_{x\to 2014\pi} \frac{-\sin(x)}{\sin(x)} \\
&= \lim_{x\to 2014\pi} -1 \\
&= -1.
\end{aligned}$$

4. Evaluate the limit $\lim_{x\to 0} \frac{\sin(x)}{1-\cos(x)}$ or show that it does not exist. Answer:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x)}{1-\cos(x)} &= \lim_{x\to 0} \frac{\sin(x)}{1-\cos(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)} \\
&= \lim_{x\to 0} \frac{\sin(x)(1+\cos(x))}{1-\cos^2(x)} \\
&= \lim_{x\to 0} \frac{\sin(x)(1+\cos(x))}{\sin^2(x)} \\
&= \lim_{x\to 0} \frac{1+\cos(x)}{\sin(x)}
\end{aligned}$$

As $x \to 0$, $1+\cos(x) \to 2$ and $\sin(x) \to 0$. As x approaches 0 from the positive side, $\sin(x) > 0$. As x approaches 0 from the negative side, $\sin(x) < 0$. This shows that

$$\begin{aligned}
\lim_{x\searrow 0} \frac{1+\cos(x)}{\sin(x)} &= \infty \\
\lim_{x\nearrow 0} \frac{1+\cos(x)}{\sin(x)} &= -\infty,
\end{aligned}$$

so the limit does not exist.

5. Evaluate the limit $\lim_{x\to 0} \frac{1-\cos(3x)}{2x^2}$ or show that it does not exist. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(3x)}{2x^2} &= \lim_{x\to 0} \frac{1-\cos(3x)}{(3x)^2} \frac{(3x)^2}{2x^2} \\ &= \frac{9}{2} \lim_{x\to 0} \frac{1-\cos(3x)}{(3x)^2} \\ &= \frac{9}{2} \cdot \frac{1}{2} = \frac{9}{4}.\end{aligned}$$

6. Evaluate the limit $\lim_{x\to 0} \frac{3x}{\sin(2x)}$ or show that it does not exist. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{3x}{\sin(2x)} &= \lim_{x\to 0} \frac{3}{2} \frac{2x}{\sin(2x)} \\ &= \frac{3}{2} \lim_{x\to 0} \left(\frac{\sin(2x)}{2x}\right)^{-1} \\ &= \frac{3}{2}.\end{aligned}$$

7. Evaluate the limit $\lim_{x\to 0} \frac{\sin(\sin(x))}{x}$. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(\sin(x))}{x} &= \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \frac{\sin(x)}{x} \\ &= \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \lim_{x\to 0} \frac{\sin(x)}{x} \\ &= 1 \cdot 1 = 1.\end{aligned}$$

8. Evaluate the limit $\lim_{x\to 0} \frac{1-\cos(\pi x)}{x^2}$. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(\pi x)}{x^2} &= \lim_{x\to 0} \frac{1-\cos(\pi x)}{pi^2 x^2} \cdot \pi^2 \\ &= \pi^2 \cdot \lim_{x\to 0} \frac{1-\cos(\pi x)}{(\pi x)^2} \\ &= \pi^2 \cdot \frac{1}{2} = \frac{\pi^2}{2}.\end{aligned}$$

9. Evaluate the limit $\lim_{x\to 0} \frac{\sin(3x^2)}{x}$ Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x^2)}{x} &= \lim_{x\to 0} \frac{\sin(3x^2)}{x} \cdot \frac{3x}{3x} \\ &= \lim_{x\to 0} \frac{\sin(3x^2)}{3x^2} \cdot 3x \\ &= 0\end{aligned}$$

10. Evaluate the limit $\lim_{x\to 0} \dfrac{\sin(x)}{x+\tan(x)}$ Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)}{x+\tan(x)} &= \lim_{x\to 0} \frac{\frac{\sin(x)}{x}}{\frac{x+\tan x}{x}} \\ &= \lim_{x\to 0} \frac{\frac{\sin(x)}{x}}{1+\frac{\sin(x)}{x\cos(x)}} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2}\end{aligned}$$

11. Evaluate the limit $\lim_{x\to 0} \dfrac{1-\cos(x)}{x\sin(x)}$ Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(x)}{x\sin(x)} &= \lim_{x\to 0} \frac{\frac{1-\cos(x)}{x^2}}{\frac{x\sin(x)}{x^2}} \\ &= \frac{1/2}{1} \\ &= \frac{1}{2}\end{aligned}$$

12. Evaluate the limit $\lim_{x\to 0} \dfrac{\sin(x)\sin(2x)}{\sin(3x)\sin(4x)}$. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)\sin(2x)}{\sin(3x)\sin(4x)} &= \lim_{x\to 0} \frac{\sin(x)}{x}\frac{\sin(2x)}{2x}\frac{3x}{\sin(3x)}\frac{4x}{\sin(4x)}\cdot\frac{1}{6} \\ &= \frac{1}{6}\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot\lim_{x\to 0}\frac{3x}{\sin(3x)}\cdot\lim_{x\to 0}\frac{4x}{\sin(4x)} \\ &= \frac{1}{6}.\end{aligned}$$

13. Evaluate the limit $\lim_{x\to 0} \dfrac{1-\cos(5x)}{\sin^2(3x)}$. Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(5x)}{\sin^2(3x)} &= \lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2}\frac{(3x)^2}{\sin^2(3x)}\frac{(5x)^2}{(3x^2)} \\ &= \frac{25}{9}\lim_{x\to 0}\frac{1-\cos(5x)}{(5x)^2}\cdot\lim_{x\to 0}\left(\frac{3x}{\sin(3x)}\right)^2 \\ &= \frac{25}{9}\cdot\frac{1}{2}\cdot 1 = \frac{25}{18}.\end{aligned}$$

14. Use the Sandwich Theorem to evaluate the limit $\lim_{x\to 0} x\cdot\sin\left(\dfrac{1}{x}\right)$. Answer: Since $-1 \leq \sin\left(\dfrac{1}{x}\right) \leq 1$ for all x, it follows that $-|x| \leq x\sin\left(\dfrac{1}{x}\right) \leq |x|$ for all x.

We know that $\lim_{x\to 0} |x| = 0$ and $\lim_{x\to 0} -|x| = 0$. Therefore, the Sandwich Theorem says that $\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$.

15. Use the definition of the derivative to find the derivative of $f(x) = 3x + 2$. Answer:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(3(x+h)+2) - (3x+2)}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

16. Determine where each of the following functions is continuous, and justify your answers:

(a) $g(x) = \begin{cases} (x^2-1)/(x+1) & \text{for } x \neq -1 \\ 2 & \text{for } x = -1 \end{cases}.$

(b) $h(x) = \begin{cases} (x^2-1)/(x+1) & \text{for } x \neq -1 \\ -2 & \text{for } x = -1 \end{cases}.$

(c) $j(x) = \begin{cases} x^2 - 2x & \text{for } |x| > 1 \\ 3x - 2 & \text{for } |x| \leq 1 \end{cases}.$

(d) $p(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$

(e) $q(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$

(f) $k(x) = \lceil x \rceil$, the "ceiling function" (which returns the smallest integer that is $\geq x$).

Answer:

(a) $g(x)$ is continuous at $x \neq -1$. At $x = -1$, $f(-1) = 2$ and

$$\begin{aligned} \lim_{x\to -1} f(x) &= \lim_{x\to -1} \frac{x^2-1}{x+1} \\ &= \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} \\ &= \lim_{x\to -1} x - 1 \\ &= -2. \end{aligned}$$

Since $\lim_{x\to -1} f(x) \neq f(-1)$, the function is discontinuous at $x = -1$.

(b) $h(x)$ is continuous at all x. Notably, at $x=-1$, $h(-1)=-2$ and

$$\begin{aligned}\lim_{x\to -1} h(x) &= \lim_{x\to -1} \frac{x^2-1}{x+1} \\ &= \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} \\ &= \lim_{x\to -1} x-1 \\ &= -2.\end{aligned}$$

Since $\lim_{x\to -1} h(x) = h(-1)$, the function is continuous at $x=-1$.

(c) $j(x)$ is continuous at $x \neq 1, -1$. At $x=-1$,

$$\begin{aligned}\lim_{x\searrow -1} j(x) &= \lim_{x\searrow -1} 3x-2 = -5 \\ \lim_{x\nearrow -1} j(x) &= \lim_{x\nearrow -1} x^2-2x = 3.\end{aligned}$$

Since $\lim_{x\searrow -1} j(x) \neq \lim_{x\nearrow -1} j(x)$, j is discontinuous at $x=-1$.
At $x=1$,

$$\begin{aligned}\lim_{x\searrow 1} j(x) &= \lim_{x\searrow 1} x^2-2 = -1 \\ \lim_{x\nearrow 1} j(x) &= \lim_{x\nearrow 1} 3x-2 = 1.\end{aligned}$$

Since $\lim_{x\searrow 1} j(x) \neq \lim_{x\nearrow 1} j(x)$, j is discontinuous at $x=1$.

(d) $p(x)$ is continuous at $x \neq 0$. At $x=0$, $\lim_{x\to 0} p(x) = \lim_{x\to 0} \sin(1/x)$ does not exist. Thus, the function is discontinuous at $x=0$.

(e) $q(x)$ is continuous at all x. Notably, at $x=0$, $q(0)=0$ and

$$\lim_{x\to 0} q(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

This follows from the Sandwich theorem and the fact that $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$. Therefore q is continuous at $x=0$.

(f) $k(x)$ is continuous all x not an integer. When $x=n$ where n is an integer,

$$\begin{aligned}\lim_{x\searrow n} k(x) &= \lim_{x\searrow n} \lceil x \rceil = n+1 \\ \lim_{x\nearrow n} fkx) &= \lim_{x\nearrow n} \lceil x \rceil = n.\end{aligned}$$

Since $\lim_{x\searrow n} k(x) \neq \lim_{x\nearrow n} k(x)$, k is discontinuous at $x=n$.

17. Find a value for a such that the function

$$f(x) = \begin{cases} \frac{6x^2-54}{x-3} & \text{for } x \neq 3 \\ a & \text{for } x = 3 \end{cases}$$

is continuous. Answer: In order for the function $f(x)$ to be continuous at $x = 3$, it needs to be the case that $\lim_{x\to 3} f(x) = f(3)$. In this case,

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{6x^2 - 54}{x-3} \\ &= \lim_{x\to 3} \frac{6(x+3)(x-3)}{x-3} \\ &= \lim_{x\to 3} 6(x+3) \\ &= 6(3+3) = 36.\end{aligned}$$

and $f(3) = a$. Therefore, $f(x)$ is continuous when $a = 36$.

18. Find all values of a so that

$$f(x) = \begin{cases} \sin(x+a) & \text{for } x < 0 \\ ax^2 & \text{for } x \geq 0 \end{cases}$$

is continuous. Answer: In order for the function $f(x)$ to be continuous at $x = 0$, it needs to be the case that $\lim_{x\searrow 0} f(x) = \lim_{x\nearrow 0} f(x) = f(0)$. In this case,

$$\begin{aligned}\lim_{x\searrow 0} f(x) &= \lim_{x\searrow 0} ax^2 \\ &= 0 \\ \lim_{x\nearrow 0} f(x) &= \lim_{x\nearrow 0} \sin(x+a) \\ &= \sin(a) \\ f(0) &= a(0)^2 = 0\end{aligned}$$

Therefore f is continuous when $\sin(a) = 0$, so when $a = \pi k$ for any integer k.

19. Find all asymptotes (horizontal, vertical, slanted) of the function $f(x) = \frac{3x^2 + x - 1}{x+3}$. Answer: $\lim_{x\searrow -3} = \infty$, so $f(x)$ has a vertical asymptote at $x = -3$.

The function $f(x)$ has no horizontal asymptotes.

Using polynomial long division, you can find that

$$\frac{3x^2 + x - 1}{x+3} = 3x - 8 + \frac{23}{x+3}.$$

Then

$$\lim_{x\to\infty} \frac{3x^2 + x - 1}{x+3} - (3x-8) = \lim_{x\to\infty} \frac{23}{x+3} = 0.$$

Therefore $f(x)$ has a slanted asymptote $y = 3x - 8$.

20. Find all asymptotes (horizontal, vertical, slanted) of the function $f(x) = \frac{x^2 - 1}{x^2 + 1}$. Answer: $\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$, so $f(x)$ has a horizontal asymptotes $y = 1$.

The function $f(x)$ has no vertical or slanted asymptotes.

21. Use the definition of the derivative to find the derivative of $g(x) = x^2$. Answer:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}$$

22. Use the definition of the derivative to find the derivative of $h(x) = \frac{2}{x}$. Answer:

$$\begin{aligned}
h'(x) &= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2x - 2(x+h)}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{-2}{(x+h)x} \\
&= -\frac{2}{x^2}
\end{aligned}$$

23. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x+1}$. Answer:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}
\end{aligned}$$