Basic Integrals, Math 221 Do as many as you can!

1. Evaluate the integral $\int 2xe^{x^2}\ dx$. Let $u = x^2$. Then $du = 2x\,dx$. Using this substitution,

$$\begin{aligned}\int 2xe^{x^2}\ dx &= \int e^u\ du\\ &= e^u + C\\ &= e^{2x} + C.\end{aligned}$$

2. Evaluate the integral $\int 15x^2\sqrt{2x^3-12}\ dx$. Let $u = 2x^3 - 12$. Then $du = 6x^2\,dx$. Using this substitution,

$$\begin{aligned}\int 15x^2\sqrt{2x^3-12}\ dx &= \int \frac{15}{6}\sqrt{u}\ du\\ &= \frac{15}{6}\frac{u^{3/2}}{3/2} + C\\ &= \frac{5}{3}(2x^3-12)^{3/2} + C.\end{aligned}$$

3. Evaluate the integral $\int \frac{1}{x\ln x}\ dx$. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$. Using this substitution,

$$\begin{aligned}\int \frac{1}{x\ln x}\ dx &= \int \frac{1}{u}\ du\\ &= \ln|u| + C\\ &= \ln|\ln(x)| + C.\end{aligned}$$

4. Evaluate the integral $\int_0^{\pi/2} \sin x\cos^5 x\ dx$. Let $u = \cos x$. Then $du = -\sin x\,dx$. At $x = 0, u = 1$. At $x = \pi/2$, $u = 0$. Using this substitution,

$$\begin{aligned}\int_0^{\pi/2} \sin x\cos^5 x\ dx &= \int_1^0 -u^5\ du\\ &= \left(\frac{-u^6}{6}\right)\Bigg|_1^0\\ &= 0 - \frac{-1}{6}\\ &= \frac{1}{6}.\end{aligned}$$

5. Evaluate the integral $\int_0^{\pi/4} \sec^2 x\tan^2 x\ dx$. Let $u = \tan x$. Then $du = \sec^2 x\,dx$. At $x = 0, u = 0$. At $x = \pi/4, u = 1$. Using this substitution,

$$\begin{aligned}\int_0^{\pi/4} \sec^2 x\tan^2 x\ dx &= \int_0^1 u^2\,du\\ &= \left(\frac{u^3}{3}\right)\Bigg|_0^1\\ &= \frac{1}{3}.\end{aligned}$$

6. Evaluate the integral $\int_0^{\pi/4} \sec^5 x \tan x \; dx$. Let $u = \sec x$. Then $du = \sec x \tan x \, dx$. We can write $\sec^5 x \tan x \, dx = \sec^4 x(\sec x \tan x \, dx) = u^4 du$. At $x = 0, u = 1$. At $x = \pi/4$, $u = \sqrt{2}$. Using this substitution,

$$\begin{aligned}\int_0^{\pi/4} \sec^5 x \tan x \; dx &= \int_1^{\sqrt{2}} u^4 \, du \\ &= \left(\frac{u^5}{5}\right)\Bigg|_1^{\sqrt{2}} \\ &= \frac{4\sqrt{2}}{5} - \frac{1}{5} = \frac{4\sqrt{2}-1}{5}.\end{aligned}$$

7. Evaluate the integral $\int x^5 \sqrt{1+x^3} \; dx$. Let $u = 1 + x^3$. Then $du = 3x^2 \, dx$. Also, note that $x^3 = u - 1$, so we can write

$$x^5 \sqrt{1+x^3} \; dx = \frac{1}{3} x^3 \sqrt{1+x^3}(3x^2 \, dx) = \frac{1}{3}(u-1)\sqrt{u} \, du.$$

Using this substitution,

$$\begin{aligned}\int x^5 \sqrt{1+x^3} \; dx &= \int \frac{1}{3}(u-1)\sqrt{u} \, du \\ &= \frac{1}{3} \int u^{3/2} - u^{1/2} \, du \\ &= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \frac{2}{15}(1+x^3)^{5/2} - \frac{2}{9}(1+x^3)^{3/2} + C.\end{aligned}$$

8. Evaluate the integral $\int \frac{10x}{5x^2-8} \; dx$. Let $u = 5x^2 - 8$. Then $du = 10x \, dx$. Using this substitution,

$$\begin{aligned}\int \frac{10x}{5x^2-8} \; dx &= \int \frac{1}{u} \, du \\ &= \ln|u| + C \\ &= \ln|5x^2 - 8| + C.\end{aligned}$$

9. Evaluate the integral $\int \frac{1}{2x+3} \; dx$. Let $u = 2x+3$. Then $du = 2 \, dx$. Using this substitution,

$$\begin{aligned}\int \frac{1}{2x+3} \; dx &= \frac{1}{2}\int \frac{1}{u} \, du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|2x+3| + C.\end{aligned}$$

10. Evaluate the integral $\int_0^1 \frac{x+1}{x^2+1}\,dx$. We can write this integral as the sum of two integrals:

$$\int_0^1 \frac{x+1}{x^2+1}\,dx = \int_0^1 \frac{x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx.$$

For the first integral, let $u = x^2+1$. Then $du = 2x\,dx$. At $x = 0, u = 1$. At $x = 1, u = 2$. Using this substitution,

$$\begin{aligned}\int_0^1 \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_1^2 \frac{1}{u}\,du \\ &= \frac{1}{2}\left(\ln|u|\right)\Big|_1^2 \\ &= \frac{1}{2}(\ln(2)-\ln(1)) = \frac{\ln(2)}{2}.\end{aligned}$$

For the second integral,

$$\begin{aligned}\int_0^1 \frac{1}{x^2+1}\,dx &= (\arctan(x))\Big|_0^1 \\ &= \arctan(1)-\arctan(0) = \frac{\pi}{4}.\end{aligned}$$

Altogether,

$$\int_0^1 \frac{x+1}{x^2+1}\,dx = \frac{\ln(2)}{2} + \frac{\pi}{4}.$$

11. Evaluate the integral $\int \frac{e^x}{e^{2x}+1}\,dx$. Let $u = e^x$. Then $du = e^x\,dx$. Using this substitution and the fact that $e^{2x} = (e^x)^2$,

$$\begin{aligned}\int \frac{e^x}{e^{2x}+1}\,dx &= \int \frac{1}{u^2+1}\,du \\ &= \arctan(u)+C \\ &= \arctan(e^x)+C.\end{aligned}$$

12. Evaluate the integral $\int e^{x+e^x}\,dx$. Let $u = e^x$. Then $du = e^x\,dx$. We can write $e^{x+e^x}\,dx = e^x e^{e^x}\,dx = e^u du$. Using this substitution,

$$\begin{aligned}\int e^{x+e^x}\,dx &= \int e^u\,du \\ &= e^u + C \\ &= e^{e^x} + C.\end{aligned}$$

13. Find the area of the finite region bounded by the curves $y = x^2$ and $y = x^3$. The region described is bounded above by $y = x^2$ and bounded below by $y = x^3$. The curves intersect at $x = 0$ and $x = 1$.

$$\begin{aligned}\text{Area} = \int_0^1 x^2 - x^3\,dx &= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= \frac{1}{12}.\end{aligned}$$

14. Find the area of the finite region bounded by the curves $y = \sqrt{x}$ and $y = \frac{x}{2}$. The region described is bounded above by $y = \sqrt{x}$ and bounded below by $y = \frac{x}{2}$. The curves intersect at $x = 0$ and $x = 4$.

$$\text{Area} = \int_0^4 \sqrt{x} - \frac{x}{2}\,dx = \left(\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right)\Big|_0^4 = \frac{4}{3}.$$

15. Find the area underneath $y = x^2 + 2$, above $y = -x$, between $x = 2$ and $x = 3$.

$$\text{Area} = \int_2^3 x^2 + 2 - (-x)\,dx = \left(\frac{x^3}{3} + 2x + \frac{x^2}{2}\right)\Big|_2^3 = \frac{65}{3}.$$

16. Find the area underneath the curve $y = x^4 - 5x^2 + 4$ that lies below the x-axis. We can factor the equation of the curve as $y = (x^2-4)(x^2-1) = (x-2)(x+2)(x-1)(x+1)$. The roots $x = 2, -2, 1, -1$ are where the curve crosses the x-axis. The curve is negative on the intervals $(-2,-1), (1,2)$, so the area can be computed in the following way:

$$\begin{aligned}\text{Area} &= -\left(\int_{-2}^{-1} x^4 - 5x^2 + 4\,dx + \int_1^2 x^4 - 5x^2 + 4\,dx\right)\\ &= -\left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right)\Big|_{-2}^{-1} - \left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right)\Big|_1^2\\ &= \frac{22}{15} + \frac{22}{15} = \frac{44}{15}.\end{aligned}$$

17. Find the area of the finite region bounded by the parabola $y^2 = x$ and the line $x + y = 6$. The region described is bounded above by $x = 6 - y$ and below by $x = y^2$. The curves intersect at $y = -3, y = 2$.

$$\text{Area} = \int_{-3}^2 (6-y) - y^2\,dy = \left(6y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-3}^2 = \frac{125}{6}.$$

18. Find the finite area bounded between $y = x^3 - 3x$ and the x-axis. The region described is bounded above by the x-axis and below by $y = x^3 - 3x$ on $(-\sqrt{3}, 0)$, and is bounded above by $y = x^3 - 3x$ and below by the x-axis on $(0, \sqrt{3})$.

$$\begin{aligned}\text{Area} &= \int_{-\sqrt{3}}^0 x^3 - 3x\,dx - \int_0^{\sqrt{3}} x^3 - 3x\,dx\\ &= \left(\frac{x^4}{4} - \frac{3}{2}x^2\right)\Big|_{-\sqrt{3}}^0 - \left(\frac{x^4}{4} - \frac{3}{2}x^2\right)\Big|_0^{\sqrt{3}}\\ &= \frac{9}{4} + \frac{9}{4}\\ &= \frac{9}{2}.\end{aligned}$$

19. Find the area above $y = |x|$ and below $y = 4$. The curve $y = |x|$ intersects $y = 4$ at $x = -4$ and $x = 4$. On the interval $(-4, 0)$, $|x| = -x$. On the interval $(0, 4)$, $|x| = x$.

$$\begin{aligned} \text{Area} &= \int_{-4}^{4} 4 - |x| \, dx \\ &= \int_{-4}^{0} 4 - (-x) \, dx + \int_{0}^{4} 4 - x \, dx \\ &= \left(4x + \frac{x^2}{2}\right)\Big|_{-4}^{0} + \left(4x - \frac{x^2}{2}\right)\Big|_{0}^{4} \\ &= 8 + 8 \\ &= 16. \end{aligned}$$

20. Find the area of the finite region between the graphs of $y = |x - 2|$ and $y = \sqrt{x}$. The region described is bounded above by $y = \sqrt{x}$ and bounded below by $y = |x - 2|$. The curves intersect at $x = 1$ and $x = 4$. On the interval $(1, 2)$, $|x - 2| = 2 - x$. On the interval $(2, 4), |x - 2| = x - 2$.

$$\begin{aligned} \text{Area} &= \int_{1}^{2} \sqrt{x} - |x - 2| \, dx \\ &= \int_{1}^{2} \sqrt{x} - (2 - x) \, dx + \int_{2}^{4} \sqrt{x} - (x - 2) \, dx \\ &= \left(\frac{2}{3}x^{3/2} - 2x + \frac{x^2}{2}\right)\Big|_{1}^{2} + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big|_{2}^{4} \\ &= \frac{8\sqrt{2} - 7}{6} + \frac{10 - 4\sqrt{2}}{3} \\ &= \frac{13}{6}. \end{aligned}$$

21. Find $\dfrac{dF}{dx}$ when $F(x) = \displaystyle\int_{0}^{x} \sin(u^2) \, du$. Let $G(u) = \displaystyle\int \sin(u^2) \, du$. By definition, $F(x) = G(x) - G(0)$.

Then $\dfrac{dF}{dx} = G'(x)$ (the $G(0)$ term vanishes because it is constant).

By the Fundamental Theorem of Calculus, $G'(x) = \sin(x^2)$. Therefore $\dfrac{dF}{dx} = \sin(x^2)$.

22. Find $\dfrac{dF}{dx}$ when $F(x) = \displaystyle\int_{0}^{x^2} t \; dt$. Let $G(t) = \int t \; dt$. By definition, $F(x) = G(x^2) - G(0)$.

By the chain rule, $\dfrac{dF}{dx} = G'(x^2) \cdot (2x)$. (The $G(0)$ term vanishes because it is constant).

By the Fundamental Theorem of Calculus, $G'(t) = t$. Therefore $\dfrac{dF}{dx} = (x^2)(2x) = 2x^3$.

23. Find $\dfrac{dF}{dx}$ when $F(x) = \displaystyle\int_{x^3}^{x^4} e^{\sin(t)} \, dt$. Let $G(t) = \int e^{\sin(t)} \, dt$. By definition, $F(x) = G(x^4) - G(x^3)$.

By the chain rule, $\frac{dF}{dx} = G'(x^4)(4x^3) - G'(x^3)(3x^2)$.

By the Fundamental Theorem of Calculus, $G'(t) = e^{\sin(t)}$. Therefore,

$$\frac{dF}{dx} = 4x^3 e^{\sin(x^4)} - 3x^3 e^{\sin(x^3)}.$$

24. Find $\frac{dF}{dx}$ when $F(x) = \int_{-x}^{x} \frac{1}{2+\cos(t)}\,dt$. Let $G(t) = \int \frac{1}{2+\cos(t)}\,dt$. By definition, $F(x) = G(x) - G(-x)$.

By the chain rule, $\frac{dF}{dx} = G'(x) - G'(-x)(-1) = G'(x) + G'(-x)$.

By the Fundamental Theorem of Calculus, $G'(t) = \frac{1}{2+\cos(t)}$. Therefore,

$$\frac{dF}{dx} = \frac{1}{2+\cos(x)} + \frac{1}{2+\cos(-x)} = \frac{2}{2+\cos(x)}.$$

The final equality holds because $\cos(x) = \cos(-x)$.