Epsilon-Delta Definition of Limits, Math 221

The $\epsilon - \delta$ definition of limits is confusing! In this worksheet, we will try to break it down and understand it better. First the definition:

" We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, we can find some $\delta > 0$ such that $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$. "

Question 1. What does the expression $|x - a| < \delta$ really mean? If you are not sure, interpret it for a concrete value of a and $\delta = .5$. Every time you see the expression $|x - a|$ you can now replace it with this phrase!

There are two other ways to interpret this definition:

Interpretation 1: A question *"If we want $f(x)$ to differ from L by no more than ϵ, then how close should x be to a?* The answer to this question is δ!

Interpretation 2: A game You are playing the Calculus Games against me. If you lose, I will point and laugh at you. I will give you a function $f(x)$, numbers a, L and a number ϵ. You have to give me back a number δ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

In both cases, **your goal given an ϵ is to give back a** δ. You are guaranteed that δ will be some expression in terms of ϵ.

Here is a general approach to doing $\epsilon - \delta$ problems:

1. Rewrite the goal.

2. Simplify $|f(x) - L|$ and rewrite it as $M|x - a|$ where M is some expression.

3. Bound M in terms of δ, normally using the triangle inequality.

4. If necessary, assume that $\delta \leq 1$ to get $|f(x) - f(a)| < c\delta$ for some constant c.

5. Now choose $\delta = \min(1, \epsilon/c)$.

Question 2: Go through the examples in the notes (pg 32-34), check that they conform to this outline.

Using this new understanding, let's revisit the limit problems done in notes. You can use the following as a model for all $\epsilon - \delta$ problems.

Question 3: Using the $\epsilon - \delta$ definition of limit show that $\lim_{x \to 5} 2x+1 = 11$.
Answer: Given $\epsilon > 0$ need to find a _______ so that if $|x-1| < \delta$ then _______ $< \epsilon$.

$$\begin{aligned} |f(x) - 11| &= |2x + 1 - 11| \\ &= \text{\_\_\_\_\_\_\_} \\ &= \text{\_\_\_\_\_\_\_} \end{aligned}$$

If $|x-5| < \delta$, then $|f(x) - 11| <$ _______. So if $\delta < \epsilon/2$ then $|f(x) - 11| <$ _______.

Question 4: Using the $\epsilon - \delta$ definition of limit show that $\lim_{x \to 3} x^2 = 9$.
Answer: Given $\epsilon > 0$ need to find a $\delta > 0$ so that if _______ then $|f(x) - 9| <$ _______.

$$\begin{aligned} |f(x) - 9| &= |x^2 - 9| \\ &= \text{\_\_\_\_\_\_\_} \end{aligned}$$

If $|x-3| < \delta$, then $|x+3| = |$_______$| = |x-3| + 6 <$ _______, so

$$|f(x) - 9| < \text{\_\_\_\_\_\_\_}$$

Assume that $\delta < 1$, so $\delta + 6 < 7$.
If $|x-3| <$ _______ and $\delta < 1$, then $|f(x) - 9| <$ _______. So if $\delta =$ min(_______, _______), then _______.

In the above examples the sentences are super important! Your solutions should have as many. Use the above as a guideline.

Question 5. Using the $\epsilon - \delta$ definition of limit show that $\lim_{x \to 1} 2x - 4 = -2$.

Question 6. Using the $\epsilon - \delta$ definition of limit show that $\lim_{x \to 4} \sqrt{x} = 2$. (Hint: to get a $|x - 4|$ in the expression for $|f(x) - 2|$ multiply and divide by an appropriate conjugate.)

Question 7. You have been promoted to chief square-builder at your factory, and federal regulations require your squares have a cross-sectional area of 100cm^2 with a maximum error of 1cm^2. Within what tolerance must you measure the side length of the square?

Question 8. Each of the following attempted definitions for when $\lim_{x \to a} f(x) = L$ is flawed: identify what the flaw is, and explain why it is a problem. "We say that $\lim_{x \to a} f(x) = L$ if....":

1. for every ϵ there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$.
2. for every $\epsilon > 0$ there is a δ such that for all x with $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$.
3. for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x with $|x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$.
4. for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x with $0 < |x - a| < \delta$, it is true that $0 < |f(x) - L| < \epsilon$.
5. for every $\delta > 0$ there is an $\epsilon > 0$ such that for all x with $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$.