# Math 320-Practice Midterm 1

September 27, 2012

1.) a.) Find general solutions to the following differential equations

i.) $xy' = \frac{y}{x}$

$$\frac{y'}{y} = \frac{1}{x^2} \rightarrow \frac{dy}{y} = \frac{dx}{x^2}$$

$$\rightarrow \ln y = -\frac{1}{x} + C$$

$$\rightarrow y = e^{-1/x + C} = A e^{-1/x}$$

ii.) $x^2y' = \frac{y}{1+\frac{1}{x}}$

$$x^2 \frac{dy}{dx} = \frac{y}{\frac{x+1}{x}} \rightarrow \frac{dy}{y} = \frac{1}{x(x+1)} dx$$

Now:

$$\int \frac{1}{x(x+1)} dx, \quad \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

so $1 = A(x+1) + Bx \rightarrow A = 1, B = -1$

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx = \ln|x| - \ln|x+1| + C$$

$$\int \frac{dy}{y} = \int \frac{1}{x(x+1)} dx \rightarrow \ln|y| = \ln|x| - \ln|x+1| + C$$

So

$$y = e^{\ln\left|\frac{x}{x+1}\right| + C} = A\left|\frac{x}{x+1}\right|$$

b.) Find the particular solutions to the following differential equations.

i.) $y' = \sqrt{xy} \quad y(0) = 1$

$$\frac{dy}{dx} = \sqrt{x}\sqrt{y} \rightarrow \frac{dy}{\sqrt{y}} = \sqrt{x}\, dx$$

$$\rightarrow \int \frac{dy}{\sqrt{y}} = \int \sqrt{x}\, dx$$

$$\rightarrow 2\sqrt{y} = \frac{2}{3} x^{3/2} + C$$

Initial Condition $y(0) = 1$

$$\rightarrow 2 = \frac{2}{3} + C \rightarrow C = \frac{4}{3}$$

$$\sqrt{y} = \frac{2}{6} x^{3/2} + \frac{4}{6}$$

$$y = \left(\frac{1}{3} x^{3/2} + \frac{2}{3}\right)^2$$

ii.) $\frac{y'}{x} = ye^x \quad y(0) = 1$

$$\frac{dy}{y} = xe^x dx \rightarrow \underbrace{\int \frac{dy}{y}}_{\ln|y|} = \int xe^x dx$$

Now

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

So $\ln|y| = xe^x - e^x + C$

Initial Condition

$$0 = e - e + C \rightarrow C = 0$$

$$\rightarrow y = \exp(xe^x - e^x)$$

2.) a.) Show that the following differential forms are exact

i.) $(6x^2y^3 + y\sin xy)dx + (9x^2y^2 + x\sin xy)dy$

$P$ and $Q$

$$\frac{\partial P}{\partial y} = 18x^2y^2 + y(x\cos xy) + \sin xy$$

$$\frac{\partial Q}{\partial x} = 18xy^2 + xy\cos xy + \sin xy$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so form is exact

ii.) $(e^x + 2y^2)dx + (4xy + \frac{1}{y})dy$

$P$ and $Q$

$$\frac{\partial P}{\partial y} = 4y$$

$$\frac{\partial Q}{\partial x} = 4y$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so form is exact.

b.)Solve the following differential equations

i.) $xyy' = x^2 + y^2$ → Homogeneous

$xyy' - y^2 = x^2 \rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{x}{y}$

$V = \frac{y}{x} \rightarrow x\frac{dv}{dx} = \frac{dy}{dx} - V$

so $\frac{dy}{dx} = V + \frac{1}{V}$ and

$x\frac{dv}{dx} = \frac{1}{V} \rightarrow \int v\,dv = \int \frac{dx}{x}$

$\rightarrow \frac{V^2}{2} = \ln x + C$

$\rightarrow \frac{y^2}{2x^2} = \ln x + C$

$\rightarrow y^2 = 2x^2\ln x + Cx^2$

ii.) $(\ln x - y\cos x + 1)dx + (2y - \sin x)dy$

$P$ and $Q$

$\frac{\partial P}{\partial y} = -\cos x$

$\frac{\partial Q}{\partial x} = -\cos x$

so exact

Assume $F(x,y)$ is a solution

$F(x,y) = \int Q\,dy = y^2 - y\sin x + g(x)$

$\frac{\partial F}{\partial x} = \cancel{-y\cos x} + g'(x) = \ln x \cancel{-y\cos x} + 1$ ($P$)

$g'(x) = \ln x + 1$

$g(x) = \int \ln x + 1\,dx = x\ln x - x + x + C$
$= x\ln x + C$

So general solution is
$x\ln x + y^2 - y\sin x = C$

3.)a.) Find the general solution to the following differential equations

$$y' + y - \cos x = -xy'$$

$y' + y + xy' = \cos x \rightarrow (x+1)y' + y = \cos x . \rightarrow y' + \frac{y}{x+1} = \frac{\cos x}{x+1}$

First order linear.

I.F.: $f(x) = e^{\int \frac{1}{x+1} dx} = e^{\ln|x+1|} = x+1$

So

$(x+1)y' + y = \cos x \rightarrow \frac{d}{dx}(x+1)y = \cos x$

$\rightarrow (x+1)y = \int \cos x \, dx \rightarrow (x+1)y = \sin x + C$

$\rightarrow y = \frac{\sin x + C}{x+1}$

b.) Find the particular solution to the following differential equations

$$y' + y = x \quad y(0) = 3$$

I.F. $f(x) = e^{\int 1 dx} = e^x$

$e^x y' + e^x y = xe^x \rightarrow \frac{d}{dx}(e^x y) = xe^x$

$\rightarrow e^x y = \int xe^x dx = xe^x - e^x + C$

$\rightarrow y = x - 1 + Ce^{-x} \qquad y(0) = 3$

$\rightarrow 3 = C - 1 \rightarrow C = 4$

$y = x - 1 + 4e^{-x}$

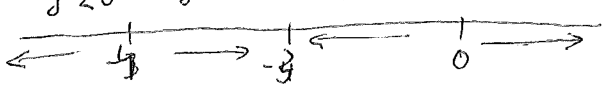
4.) a.) Draw the phase diagrams for the following differential equations identify the critical points, and determine their stability.

i.) $y' = y^3 + 7y^2 + 12y$

$y^3 + 7y^2 + 12y = 0 \rightarrow y(y+3)(y+4) = 0$

$y = 0, -3, -4$ critical points

$y' < 0$ $\quad y' > 0$ $\quad y' < 0$ $\quad y' > 0$




So $y = -3$ is stable

$y = -4, 0$ are unstable.

ii.) $y' = y^2 - y - 6$

$y^2 - y - 6 = 0 \rightarrow (y-3)(y+2) = 0$

Critical points $y = 3$, $y = -2$

$y' > 0$ $\quad y' < 0$ $\quad y' > 0$


So $y = -2$ is stable

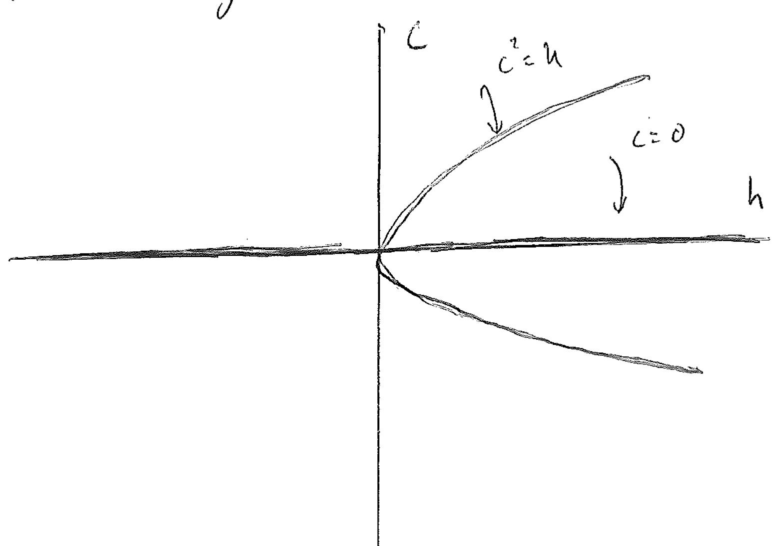
$y = 3$ is unstable

b.) Draw the bifurcation diagram for the following differential equations

$y' = hy - y^3$

$hy - y^3 = 0 \rightarrow y(h - y^2) = 0 \rightarrow y = 0, \; y = \pm\sqrt{h}$

Bifurcation Diagram $\quad c = \pm\sqrt{h} \rightarrow c^2 = h$



5.) Consider the following differential equation $y' = x^2+1$ with initial condition $y(0) = 1$. Let $h = 1$ and estimate the value of $y(2)$ using both the Euler Method and the Improved Euler Method

Euler's Method

| $y \backslash x$ | 0 | 1 | 2 |
|---|---|---|---|
| $y$ | 1 | $1+1\cdot(1^2+1)$ $= 2$ | $2+1\cdot(2^2+1)$ $= 7$ |

Improved Euler

| | 0 | 1 | 2 |
|---|---|---|---|
| $k_1$ | | $0^2+1 = 1$ | $1^2+1 = 2$ |
| $u$ | | $1+1\cdot 1 = 2$ | $\frac{5}{2}+1\cdot 2 = \frac{9}{2}$ |
| $k_2$ | | $1^2+1 = 2$ | $5$ |
| $y$ | 1 | $1+1\cdot\frac{1}{2}\cdot 3 = \frac{5}{2}$ | $\frac{5}{2}+\frac{7}{2} = 6$ |

$k_1 = f(x_n, y_n)$

$u_{n+1} = y_n + h k_1$

$k_2 = f(x_{n+1}, u_{n+1})$

$y_{n+1} = y_n + h \cdot \frac{1}{2}(k_1+k_2)$

* → These use formula from book

6.) The equation

$$t^2 dx - (x + \sin t)dt = 0$$

with $x(0) = 1$ does not have a solution. Show that it does not ahve a solution and explain why this does not contradict the existence theorem for solutions of differential equations.

$P = t^2,$ $\quad Q = -(x + \sin t)$

$\frac{\partial P}{\partial t} = 2t$ $\quad \frac{\partial Q}{\partial x} = -1$

Since this is not exact, there is no solution

Now $\frac{dx}{dt} = \frac{x + \sin t}{t^2}.$ Since this equation is not continuous at $t = 0$, we can't use the existence theorem to conclude there is a solution.

7.)Consider the differential equation

$$y' = x - y$$

with initial condition $y(0) = 1$.Use the Runge-Kutta method with $h = 1$ to estimate $y(2)$

$\rightarrow$ $K_1 = f(x_n, y_n) = -1$, $K_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1) = f(\frac{1}{2}, \frac{1}{2}) = 0$

$K_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_2 h) = f(\frac{1}{2}, 1 + \frac{1}{2}\cdot 0) = -\frac{1}{2}$

$K_4 = f(x_{n+1}, y_n + hK_3) = \frac{1}{2}$

$y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + \frac{1}{6}(-1 + 0 - 1 + \frac{1}{2}) = \frac{9}{12} = .75$

$\rightarrow$ $K_1 = f(1, \frac{3}{4}) = \frac{1}{4}$, $K_2 = f(\frac{3}{2}, \frac{3}{4} + \frac{1}{8}) = \frac{5}{8}$

$K_3 = f(\frac{3}{2}, \frac{3}{4} + \frac{5}{16}) = \frac{7}{16}$, $K_4 = f(2, \frac{3}{4} + \frac{7}{16}) = -\frac{3}{16}$

$y_2 = \frac{3}{4} + \frac{1}{6}\left(\frac{1}{4} + 2\cdot\frac{5}{8} + 2\cdot\frac{7}{16} - \frac{3}{16}\right)$

$= \frac{107}{96}$

8.) Put the following matrices in Row Echelon Form

a.) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{Swap}} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow{-2R_1+R_2}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

b.) $\begin{pmatrix} -1 & 0 & 4 \\ 2 & -4 & 6 \\ 3 & 4 & 12 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 & 4 \\ 2 & -4 & 6 \\ 3 & 4 & 12 \end{pmatrix} \xrightarrow{2R_1+R_2} \begin{pmatrix} -1 & 0 & 4 \\ 0 & -4 & 14 \\ 3 & 4 & 12 \end{pmatrix} \xrightarrow{3R_1+R_2} \begin{pmatrix} -1 & 0 & 4 \\ 0 & -4 & 14 \\ 0 & 4 & 26 \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} -1 & 0 & 4 \\ 0 & -4 & 14 \\ 0 & 0 & 40 \end{pmatrix}$$

c.) $\begin{pmatrix} 1 & 6 & -1 \\ 2 & 5 & 1 \\ 0 & 2 & 1 \\ -4 & -17 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 6 & -1 \\ 2 & 5 & 1 \\ 0 & 2 & 1 \\ -4 & -17 & 1 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 6 & -1 \\ 0 & -7 & 3 \\ 0 & 2 & 1 \\ -4 & -17 & 1 \end{pmatrix} \xrightarrow{4R_1+R_4} \begin{pmatrix} 1 & 6 & -1 \\ 0 & -7 & 3 \\ 0 & 2 & 1 \\ 0 & 7 & -3 \end{pmatrix} \xrightarrow{R_2+R_4} \begin{pmatrix} 1 & 6 & -1 \\ 0 & -7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{2}{7}R_2+R_3} \begin{pmatrix} 1 & 6 & -1 \\ 0 & -7 & 3 \\ 0 & 0 & \frac{13}{7} \\ 0 & 0 & 0 \end{pmatrix}$$