Math 567 - Practive Final Exam:

December 12, 2012

1.) Solve the given differential equation with initial condition

a.) $y' = \frac{1}{x^4}$ $y(3) = -2$

$$y = \frac{-1}{3x^3} - \frac{49}{24}$$

b.) $\frac{dy}{dx} = 2x\sqrt{y}$ $y(2) = 3$

$$y = \left(\frac{x^2}{2} - \sqrt{3} - 2\right)^2$$

c.) $y' - y = xy + 1$ $y(2) = 0$

I used integration factors but got stuck...

2.)a.) Show the following equation is exact and then solve it

$$(\overset{P}{\overset{\shortparallel}{3x^2 - \sqrt{y}}})dx + \overset{Q}{\overset{\shortparallel}{\left(-\frac{x}{2\sqrt{y}}\right)}}\, dy = 0$$

$\frac{\partial P}{\partial x} = \frac{-1}{2\sqrt{y}}$ $\quad \frac{\partial Q}{\partial y} = \frac{-1}{2\sqrt{y}}$ $\quad$ so $\quad F(x,y) = \int 3x^2 - \sqrt{y}\, dx = x^3 - x\sqrt{y} + g(y)$

$\frac{\partial F}{\partial y} = \frac{-x}{2\sqrt{y}} + g'(y) = \overset{Q}{\downarrow}\frac{-x}{2\sqrt{y}} \quad \rightarrow \quad g'(y) = 0 \rightarrow g(y) = C$

$\circ\ F(x,y) = x^3 - x\sqrt{y} + C$

b.) Solve the following equation

$$xy^2y' = x^3 + y^3 =$$

This is homogeneous. Let $v = \frac{y}{x}$

$y' = \frac{x^2}{y^2} + \frac{y}{x} = \frac{1}{v^2} + v$

Now $\quad x\frac{dv}{dx} = y' - v = \frac{1}{v^2} + v - v = \frac{1}{v^2}$

$\rightarrow \quad v^2\, dv = \frac{dx}{x} \quad \rightarrow \quad \frac{v^3}{3} = \ln x + C$

so $\quad \frac{y^3}{x^3} = 3\ln x + C \quad \rightarrow \quad y^3 = 3x^3\ln x + Cx^3$

3.) Solve the following equations for a general solution, determine the critical points, their stability and draw the bifurcation diagram.
$\frac{dx}{dt} = 5x(7-x)$

I'm not sure a bifurcation diagram makes sense

General solution can be got by solving:

$$\frac{dx}{5x(7-x)} = dt$$

Alternatively, this is a population model - See pg 82.

$$x(t) = \frac{7C}{C + (2)e^{-35t}}$$

Crit Points

$5x(7-x) = 0 \qquad x = 0, 7$

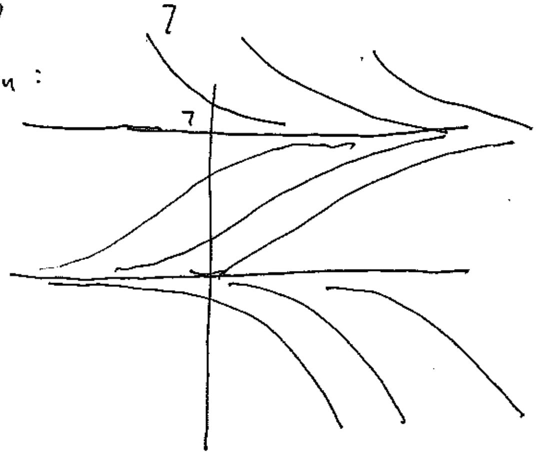
$x = 7 \rightarrow$ stable
$x = 0 \rightarrow$ unstable

Phase Diagram:

$x' < 0 \qquad x' > 0 \qquad x' < 0$

$0 \qquad 7$

General Solution:



4.)Consider the equation

$$y' = 2x - y$$

Consider the solution curve starting at $y(1) = 2$. Let $h = 2$ and use each of the three methods that we learned to estimate the valie of $y(3)$

6.) Find the inverse of the following matrices and use them to find the solution to the vector equations $Ax = b$

i.) $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $A^{-1}b = \begin{bmatrix} 3/2 \\ -1 \\ 0 \end{bmatrix}$

ii.) $\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 0 & 1 \end{bmatrix}$ $A^{-1}b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

7.) Calculate the determinants of the following matrices.

a.) $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

$\det A = 14$

c.) $\begin{array}{c} + \\ - \\ + \\ - \end{array}\begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -5 \\ -2 & 1 & 0 & 1 \end{bmatrix}$

We can take cofactors on any row or column, as long as we are careful about signs

$$\det A = -(-2)\begin{array}{c}\begin{vmatrix} 1 & 1 & 4 \\ 1 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} \\ \begin{matrix} + & - & + \end{matrix}\end{array} = -(-2)\left(-5\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -10\,(3-1) = -20$$

I present 2 different ways.

8.) Determine if the following sets of vectors are linearly independent.

a.) $\left\{ \begin{array}{c} (0,2,-1) \\ (9,1,3) \\ (3,2,5) \end{array} \right\}$

$$\det \begin{pmatrix} 0 & 2 & -1 \\ 9 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix} = -87 \neq 0 \quad \text{so} \quad \text{lin ind.}$$

c.) $\left\{ \begin{array}{c} (1,2,-1,-5) \\ (1,-1,1,-2) \\ (-2,5,-1,3) \\ (-1,2,2,10) \end{array} \right\} \begin{array}{l} = v_1 \\ = v_2 \\ = v_3 \\ = v_4 \end{array}$

Need $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$\rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0$

v_i's go in as columns

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 2 & -1 & 5 & 2 & 0 \\ -1 & 1 & -1 & 2 & 0 \\ -5 & -2 & 3 & 10 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & -3 & 9 & 4 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 3 & -7 & 5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & -3 & 9 & 4 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & 1 & -6 & -5 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 9 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & 1 & -6 & -5 & 0 \\ 0 & 0 & 9 & -9 & 0 \\ 0 & 0 & 2 & 9 & 0 \end{array}\right) \rightarrow \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & 1 & -6 & -5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$c_4 = 0$

$c_3 - c_4 = 0 \rightarrow c_3 = 0$

$c_2 + 6c_3 - 5c_4 = 0 \rightarrow c_2 = 0$

$c_1 + c_2 - 2c_3 - c_4 = 0 \rightarrow c_1 = 0$

9.) Find the general solution to the following second order differential equations.

i.) $y'' + 17y' + 70y = 0$

$r^2 + 17r + 70 = 0 \quad \rightarrow \quad (r+7)(r+10) = 0$

$y = C_1 e^{-7x} + C_2 e^{-10x}$

ii.) $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0 \quad (r+3)^2 = 0 \quad r = -3$ w/ mult 2

$y = C_1 e^{-3x} + C_2 x e^{-3x}$

10.) Find a vector $\mathbf{v}$ in $\mathbb{R}^4$ that is not in the span of the vectors $\mathbf{w}$ and $\mathbf{x}$.

i.)$\mathbf{w}=(2,-2,1,0)$ $\mathbf{x}=(0,0,1,-1)$

I claim $v=(1,0,0,0)$ is not in span $\{w,x\}$

$c_1 w + c_2 x = v$

$$\left(\begin{array}{cc|c} 2 & 0 & 1 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right)$$

inconsistent so no such c_1, c_2

ii.) $\mathbf{w}=(1,2,-1,-2)$ $\mathbf{x}=(-1,5,1,3)$

Again $v=(1,0,0,0)$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 5 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 0 \end{array}\right) \xrightarrow{R_1+R_3} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{array}\right)$$

inconsistent

11.) Find a basis for the indicated spaces.

a.) The set of all (x, y, z) such that $3x - y = 2z$

$z = s \quad , \quad y = t \qquad 3x = 2s + t \rightarrow x = \dfrac{2s+t}{3}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}$$

Basis : $\left\{ \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$

b.) The nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ 2 & 5 & 0 \end{bmatrix}$$

Nullspace : $\{ x \in \mathbb{R}^3 \mid Ax = 0 \}$ i.e. all x s.t $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\quad x + y + z = 0 \qquad z = s \rightarrow \quad y = \dfrac{-2z}{3} = \dfrac{-2}{3}s$

$\quad 3y + 2z = 0$

$\qquad x = \dfrac{-1}{3}s$

Basis : $\left\{ \begin{bmatrix} -1/3 \\ -2/3 \\ 1 \end{bmatrix} \right\}$

12.) Given matrices **A** and **B**. Find a matrix **E** such that **EA** = **B**

a.) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

Method 1

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \\ 2 & 0 & -4 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ & & \end{pmatrix} \xrightarrow{\frac{1}{2}R_3+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Method 2

B is

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & -4 \end{pmatrix} \begin{matrix} \to R_1 \\ \to R_1 + R_2 + R_3 \\ \to 2R_3 \end{matrix}$$

So $E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

The book does this differently. I am doing the method from lecture

13.)a.)Solving the following matrix differential equations **without** using matrix expontiation.

$$\mathbf{X} = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} \mathbf{X} \qquad \mathbf{X}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Char poly: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow (\lambda+1)^2 = 0$

$$A - (-1)I = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

$(A+I)v_1 = 0$

$$\left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$(A+I)v_2 = v_1$

$$\left(\begin{array}{cc|c} 0 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right) \rightarrow v_2 = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \right) e^{-t}$$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \rightarrow \begin{array}{l} c_2 = 3 \\ c_1 + c_2 = 3 \rightarrow c_1 = 0 \end{array}$$

$$\vec{x}(t) = 0 \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 3t + 3 \\ 1 \end{bmatrix} e^{-t} = \cancel{\begin{bmatrix} (3t+3)e^{-t} \\ e^{-t} \end{bmatrix}} = \begin{bmatrix} 3t+3 \\ 1 \end{bmatrix} e^{-t}$$

b.) $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}\mathbf{X}$ $\qquad\qquad$ $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\lambda^2 - \lambda - 6 = 0 \rightarrow (\lambda - 3)(\lambda + 2) = 0 \rightarrow \lambda = 3, \lambda = -2$

$\underline{\lambda = 3}$ $\quad \left(\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right) \qquad V = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\underline{\lambda = -2}$ $\quad \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right) \qquad V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{X}(t) = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

$$X(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ -2 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 1 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -6 \end{array}\right]$$

$c_1 = 2, \quad c_2 = -6$

14.) Solving the following matrix differential equation using matrix expontiation.

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{X} \qquad \mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<u>Steps</u>

1) Find evectors
2) Find P, P^{-1}
3) Find D
4) Compute $\vec{x}(t) = P e^{Dt} P^{-1} X(0)$

$$\rightarrow \quad \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0 \qquad \lambda = 3, -1$$

<u>$\lambda = 3$</u>

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \rightarrow V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>$\lambda = -1$</u>

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \rightarrow V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad P^{-1} = \frac{-1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\vec{X}(t) = P e^{Dt} P^{-1} X(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}} \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & e^{-t} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$