Undetermined Coefficients

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We are interested in solving differential equations of the type

$$y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = f(x).$$

When presented with such a non-homogeneous differential equation, we have to take a good guess at a particular solution and then plug this back in to determine the coefficients. The goal of this document is to help you take good guesses.

Rule 1: If the complementary solution of the homogenous equation does not have any terms in common with $f(x)$ or it's derivatives, then take a linear combination of the terms in $f(x)$ and its derivatives.

Example 1. • $y'' - y = e^{2x}$

- It's easy to see that $y_c = c_1 e^x + c_2 e^{-x}$
- Then $y_p = Ae^{2x}$

Example 2. • $y'' - 3y' + 2y = 3e^{-x} - 10x\cos x$

- It's easy to see that $y_c = c_1 e^x + c_2 e^{2x}$
- Then $y_p = Ae^{-x} + Bx\cos x + Cx\sin x$

Example 3. • $y''' + 9y' = x\sin x + x^2 e^{2x}$

- It's easy to see that $y_c = c_1 + c_2\cos 3x + c_3\sin 3x$
- Then $y_p = (A + Bx + Cx^2)e^{2x} + Bx\cos x + Cx\sin x + D\sin x + E\cos x$

Note that you can start guessing a general pattern from these. I encourage you to make a list of what to guess when various functions show up.

Time for another rule.

Rule 2: When the solution to the homogenous equation is not independent of the terms in $f(x)$ and it's derivatives. Do the following:

- **Compute y_p as in rule 1.**
- **Multiply any terms that overlap with the complementary solution with a high enough power of x so that no terms are duplicated between y_c and y_p.**

This is best understood through examples.

Example 4. • $y''' + y'' = 3e^x + 4x^2$

- It's easy to see that $y_c = c_1 + c_2 x + c_3 e^{-x}$
- Our guess is that $y_p = (A + Bx + Cx^2) + De^{-x}$
- This has overlap with y_c, $A + Bx + Cx^2$ overlaps $c_1 + c_2 x$, so we multiply by a high enough power of x.
- $y_p = x^2(A + Bx + Cx^2) + xe^x$.

Example 5. • $y'' + 6y' + 13y = e^{-3x} \cos 2x$

- It's easy to see that $y_c = e^{-3x}(c_1 \cos 2x + c_2 sin 2x)$
- Our guess is that $y_p = Ae^{-3x} \cos 2x + Be^{-3x} \sin 2x$
- This has overlap with y_c, so we multiply by a high enough power of x
- $y_p = Axe^{-3x} \cos 2x + Bxe^{-3x} \sin 2x$.

Example 6. • $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$

- It's easy to see that $y_c = Ae^x + Be^{-x} + (C + Dx + Ex^2)$
- Our guess is that $y_p = Ae^x + (B + Cx + Dx^2)$
- This has overlap with y_c, so we multiply by a high enough power of x
- $y_p = Axe^x + x^3(B + Cx + Dx^2)$.

There is a helpful table on pg 346.