Undetermined Coefficients

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We are interested in solving differential equations of the type

\[ y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1y' + a_0y = f(x). \]

When presented with such a non-homogeneous differential equation, we have to take a good guess at a particular solution and then plug this back in to determine the coefficients. The goal of this document is to help you take good guesses.

**Rule 1:** If the complementary solution of the homogenous equation does not have any terms in common with \( f(x) \) or its derivatives, then take a linear combination of the terms in \( f(x) \) and its derivatives.

**Example 1.**
- \( y'' - y = e^{2x} \)
  - It’s easy to see that \( y_c = c_1 e^x + c_2 e^{-x} \)
  - Then \( y_p = Ae^{2x} \)

**Example 2.**
- \( y'' - 3y' + 2y = 3e^{-x} - 10x \cos x \)
  - It’s easy to see that \( y_c = c_1 e^x + c_2 e^{2x} \)
  - Then \( y_p = Ae^{-x} + Bx \cos x + Cx \sin x \)

**Example 3.**
- \( y''' + 9y' = x \sin x + x^2 e^{2x} \)
  - It’s easy to see that \( y_c = c_1 + c_2 \cos 3x + c_3 \sin 3x \)
  - Then \( y_p = (A + Bx + Cx^2)e^{2x} + Bx \cos x + Cx \sin x + D \sin x + E \cos x \)

Note that you can start guessing a general pattern from these. I encourage you to make a list of what to guess when various functions show up.

Time for another rule.

**Rule 2:** When the solution to the homogenous equation is not independent of the terms in \( f(x) \) and its derivatives. Do the following:
- Compute \( y_p \) as in rule 1.
- Multiply any terms that overlap with the complementary solution with a high enough power of \( x \) so that no terms are duplicated between \( y_c \) and \( y_p \).

This is best understood through examples.
Example 4. \( y''' + y'' = 3e^x + 4x^2 \)

- It’s easy to see that \( y_c = c_1 + c_2x + c_3e^{-x} \)
- Our guess is that \( y_p = (A + Bx + Cx^2) + De^{-x} \)
- This has overlap with \( y_c, A + Bx + Cx^2 \) overlaps \( c_1 + c_2x \), so we multiply by a high enough power of \( x \).
- \( y_p = x^2(A + Bx + Cx^2) + xe^x. \)

Example 5. \( y'' + 6y' + 13y = e^{-3x} \cos 2x \)

- It’s easy to see that \( y_c = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x) \)
- Our guess is that \( y_p = Ae^{-3x} \cos 2x + Be^{-3x} \sin 2x \)
- This has overlap with \( y_c \), so we multiply by a high enough power of \( x \)
- \( y_p = Axe^{-3x} \cos 2x + Bxe^{-3x} \sin 2x. \)

Example 6. \( y^{(5)} - y^{(3)} = e^x + 2x^2 - 5 \)

- It’s easy to see that \( y_c = Ae^x + Be^{-x} + (C + Dx + Ex^2) \)
- Our guess is that \( y_p = Ae^x + (B + Cx + Dx^2) \)
- This has overlap with \( y_c \), so we multiply by a high enough power of \( x \)
- \( y_p = Axe^x + x^3(B + Cx + Dx^2). \)

There is a helpful table on pg 346.