Rings in Sage

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In the following, I will try to give a detailed example of how to work with rings/fields in Sage. Most of what I will do is taken from the following Sage references: `http://www.sagemath.org/doc/tutorial/`, `http://www.sagemath.org/doc/constructions/rings.html`. You are required to know most of the features in this tutorial. As we go through this document, I would recommend typing the commands as they are done.

Basic Rings

Sage has several basic rings already implemented. Almost every ring we use in this course is probably in Sage.

```
sage: QQ
Rational Field
sage: 1/2 in QQ
True
sage: sqrt(2) in QQ
False
sage: i^2
-1
sage: parent(3.4)
Real Field with 53 bits of precision
sage: 3.4 in QQ
True
```

The last command is an example of *coercion*, a fancy word for realizing that 3.4 is an element of $\mathbb{Q}$ a subfield of $\mathbb{R}$.

Exercise. What is the command for the field for $\mathbb{R}$ and $\mathbb{C}$? Is the $\sqrt{2} \in \mathbb{R}$ according to Sage?

The command `parent` tells you what ring an element belongs to. We can also construct rings that are common in number theory.

```
sage: GF(7)
Finite Field of order 7
sage: GF(27, 'a')
Finite Field in 'a' of size 3^3
sage: Integers(5)
Ring of integers mod 5
```

We can do arithmetic in these rings.

```
sage: R = Integers(8)
sage: list(R)
sage: a = R(3); b = R(5)
sage: a*b
7
sage: F.<k> = GF(49,'k')
sage: k^3
k+2
```

Exercise. If you press Tab after a variable on the command line, you will be given a list of commands you an run on that variable. If you put a ? in front of a command, you get help on it.

- How would you determine the defining polynomial of F?
- What is the multplicative order of k?

We can also construct basic polynomial rings.

```
sage: S= QQ['t']
sage: R.<t> = PolynomialRing(QQ)
sage: U = PolynomialRing(QQ,'t')
sage: p = (t-5)*(t-7)
t^2-12*t+35
sage: factor(t^2+2*t+1)
(t+1)^2
```

Ideals and Quotients

To create an ideal, you can just specify generators.

```
sage: S = QQ['x, y']
sage: I = S.ideal([x,y])
sage: x^2+y^2 in I
True
sage: I^2
Ideal (x^2,x*y, y*x, y^2) of Multivariate Polynomial Ring in x, y over Rational Field
```

Exercise. Show that the ideal $(x, y) = (x, x + y)$ using Sage.

We can also create quotient rings.

```
sage: T = S.quotient(S.ideal([x^3]))
sage: xbar,ybar = T.gens()
```

Exercise. Read up on the QuotientRing command. `http://www.sagemath.org/doc/reference/sage/rings/quotient.`