1.) a.) Find the inverse of the following matrices.

i.) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & -5 & 1 & -3 & -2 \end{array}\right]$$

$$\xrightarrow[R_3/5]{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & -2 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1/5 & 3/5 & 2/5 \end{array}\right] \xrightarrow[3R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & -1/5 & 4/5 \\ 0 & 1 & 0 & 2/5 & -1/5 & 1/5 \\ 0 & 0 & 1 & -1/5 & 3/5 & 2/5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2/5 & -1/5 & 4/5 \\ 2/5 & -1/5 & 1/5 \\ -1/5 & 3/5 & 2/5 \end{bmatrix}$$

ii.) $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -5 & -5 & -3 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{5R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 5 & -3 & 5 & -2 \end{array}\right] \xrightarrow{R_3/5} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3/5 & 1 & -2/5 \end{array}\right]$$

$$\xrightarrow[-2R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11/5 & -2 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1 & 4/5 \\ 0 & 0 & 1 & -3/5 & 1 & -2/5 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/5 & 0 & 1/5 \\ 0 & 1 & 0 & 6/5 & -1 & 4/5 \\ 0 & 0 & 1 & -3/5 & 1 & -2/5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/5 & 0 & 1/5 \\ 6/5 & -1 & 4/5 \\ -3/5 & 1 & -2/5 \end{bmatrix}$$

b.) For each of the matrices in part a.) find the solution to the vector equations $Ax = b$ (i.e. there should be a total of 4 solutions)

Note: $Ax = b \rightarrow x = A^{-1}b$ if A is invertible

i.) $b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 2/5 & -1/5 & 4/5 \\ 2/5 & -1/5 & 1/5 \\ -1/5 & 3/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/5 & 0 & 1/5 \\ 6/5 & -1 & 4/5 \\ -3/5 & 1 & -2/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/5 \\ 4/5 \end{bmatrix}$$

ii.) $b = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 2/5 & -1/5 & 4/5 \\ 2/5 & -1/5 & 1/5 \\ -1/5 & 3/5 & 2/5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} +2 \\ -2/5 \\ +11/5 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 2/5 & -1/5 & 4/5 \\ 2/5 & -1/5 & 1/5 \\ -1/5 & 3/5 & 2/5 \end{bmatrix}}$$

$$\begin{bmatrix} -1/5 & 0 & 1/5 \\ 6/5 & -1 & 4/5 \\ -3/5 & 1 & -2/5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/5 \\ -2/5 \\ 1/5 \end{bmatrix}$$

2.) Calculate the determinants of the following matrices.

a.) $\begin{bmatrix} -1 & 3 & -1 \\ 2 & 5 & 10 \\ 1 & 19 & 17 \end{bmatrix}$

$$-1\begin{vmatrix} 5 & 10 \\ 19 & 17 \end{vmatrix} - 3\begin{vmatrix} 2 & 10 \\ 1 & 17 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 1 & 19 \end{vmatrix}$$

$$= -1(85-190) - 3(34-10) - 1(38-5)$$

$$= 105 - 72 - 33 = 0$$

b.) $\begin{bmatrix} 7 & 2 & -3 \\ 6 & -2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

$$7\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 6 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= 7(-4) - 2(0) - 3(12)$$

$$= -64$$

c.) $\begin{bmatrix} 0 & 2 & 1 & 4 \\ 3 & 2 & -4 & 1 \\ -3 & 2 & 1 & -5 \\ -2 & 1 & 1 & -4 \end{bmatrix}$ Simplify problem $\rightarrow$ $R_2 + R_3$

$$\rightarrow \det\begin{bmatrix} 0 & 2 & 1 & 4 \\ 3 & 2 & -4 & 1 \\ 0 & 4 & -3 & -4 \\ -2 & 1 & 1 & -4 \end{bmatrix} = -3\begin{vmatrix} 2 & 1 & 4 \\ 4 & -3 & -4 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & 4 \\ 2 & -4 & 1 \\ 4 & -3 & -4 \end{vmatrix}$$

$$= -3\left(2\begin{vmatrix} -3 & -4 \\ 1 & -4 \end{vmatrix} - 1\begin{vmatrix} 4 & -4 \\ 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 4 & -3 \\ 1 & 1 \end{vmatrix}\right)$$

$$+2\left(2\begin{vmatrix} -4 & 1 \\ -3 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 4 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & -4 \\ 4 & -3 \end{vmatrix}\right)$$

$$= -3(32 + 12 + 28) + 2(38 + 12 + 40)$$

$$= -216 + 180 = -36.$$

3.) Determine if the following sets of vectors are linearly independent.

a.) $\left\{ \begin{matrix} (1,3,-1) \\ (0,1,2) \\ (-2,4,-1) \end{matrix} \right\}$

$$\det \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ -2 & 4 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = -7 - 14 = -21$$

$\det \neq 0$ so lin ind

b.) $\left\{ \begin{matrix} (1,3,-1) \\ (0,1,2) \\ (-2,4,-1) \\ (-1,0,1) \end{matrix} \right\}$

4 vectors in $\mathbb{R}^3$ are linearly independent.

c.) $\left\{ \begin{matrix} (1,3,-1,0) \\ (0,1,2,-2) \\ (-2,4,-1,3) \\ (-1,2,2,-3) \end{matrix} \right\}$

$$\det \begin{pmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & -2 \\ -2 & 4 & -1 & 3 \\ -1 & 2 & 2 & -3 \end{pmatrix} = 46 \neq 0$$

so lin ind.

4.) What is the dimension of the following vector spaces? (you MUST justify your answer).

a.) The set of all vectors in $\mathbb{R}^5$ of the form $v = (2a + 3b, c - 2a, 0, a + c, b - c)$

$(2a+3b, c-2a, 0, a+c, b-c) = a(2, -2, 0, 1, 0) + b(3, 0, 0, 0, 1) + c(0, 1, 0, 1, -1)$

Dimension is 3 since these vectors are lin independent*

* If this not obvious to you ... CHECK.

b.) The set of all functions f such that $f'(x) = 0$.

$f'(x) = 0 \Rightarrow f(x)$ is a constant. AKA. $f(x) = k \cdot 1$

so dimension is 1

c.)The set of all $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \\ z \end{bmatrix}$ such that $A\mathbf{x} = 0$, where $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -2 & -3 & -4 \\ 0 & 0 & -2 & -4 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ -1 & -2 & -3 & -4 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right] \xrightarrow[R_1 + R_3]{R_2 + R_4} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 6 & -5 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z = 0$

$w - \frac{1}{2}z = 0$

$x + 2y + 3w - z = 0$

$\longrightarrow$

$z = 0$

$w = \frac{1}{2} \cdot 0 = 0$

$x = -2y$

Basis for solution space $(-2, 1, 0, 0)$ so dimension is 1.

5.) a.) Find the general solution to the following second order differential equations.

i.) $y'' + 4y' + 3y = 0$

$r^2 + 4r + 3 = 0 \rightarrow (r+3)(r+1) = 0$

$\rightarrow r = -3, -1$

$\rightarrow y(x) = c_1 e^{-3x} + c_2 e^{-x}$ is general solution

ii.) $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0 \rightarrow (r+2)^2 = 0$

$y(x) = (c_1 + c_2 x) e^{-2x}$ is general solution.

b.)Find the particular solution to the following initial value problems.

i.) $y'' + 2y' - 3y = 0$ With $y'(0) = 1$ and $y(0) = 2$

$r^2 + 2r - 3 = 0 \rightarrow (r-1)(r+3) = 0 \rightarrow r = 1, -3$

$y(x) = c_1 e^x + c_2 e^{-3x}$, $y' = c_1 e^x - 3c_2 e^{-3x}$

$y(0) = c_1 + c_2 = 2$

$y'(0) = c_1 - 3c_2 = 1$

$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -4 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/4 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7/4 \\ 0 & 1 & 1/4 \end{array}\right] \rightarrow \begin{array}{l} c_1 = 7/4 \\ c_2 = 1/4 \end{array}$

$\boxed{y(x) = \frac{7}{4} e^{x} + \frac{1}{4} e^{-3x}}$

ii.) $y'' + -6y' + 9y = 0$ With $y'(0) = 2$ and $y(0) = -1$

$r^2 - 6r + 9 = (r-3)^2 = 0 \rightarrow r = 3$

$y(x) = (c_1 + c_2 x) e^{3x}$

$y'(x) = 3c_1 e^{3x} + c_2(e^{3x} + 3x e^{3x})$

$y(0) = c_1 = -1$

$y'(0) = 3c_1 + c_2 = 2$

$\rightarrow c_1 = -1, \quad c_2 = 2 + 3 = 5$

$\boxed{y(x) = (-1 + 5x) e^{3x}}$

6.) Determine if the vector **v** is in the span of the vectors **w** and **x**.

i.) **v**= (1, 0, 0) **w**= $(\frac{1}{3}, -2, 1)$ **x**= $(\pi, 0, 1)$

$c_1 w + c_2 x = v$

$$\left(\begin{array}{cc|c} 1/3 & \pi & 1 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow[\text{steps}]{\text{(I'm skipping)}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3\pi & 3 \end{array}\right) \xrightarrow[-R_1+R_3]{-R_1+R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3\pi & 3 \end{array}\right)$$

$$\xrightarrow{-3\pi R_2+R_3} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right)$$

inconsistent

$V \notin \text{span}\{w, x\}$

ii.) **v**= $(0, \frac{3}{2}, \frac{1}{2})$ **w**= (1, 2, 3) **x**= (−1, 1, −2)

$c_1 w + c_2 x = v$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3/2 \\ 3 & 2 & 1/2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 3/2 \\ 0 & 1 & 1/2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \quad \cancel{W} = \frac{1}{2} w + \frac{1}{2} x$$

$V \in \text{span}\{w, x\}$

7.) Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be vectors in a vector space V.

a.) Define what it means for these vectors to form a basis for V.

b.) What is the dimension of V?

8.) Find a basis for the indicated spaces.

a.) The set of all (x, y, z) such that $2x + 4y = z$

$x = s$, $y = t$ general solution: $(s, t, 2s+4t)$

$(s, t, 2s+4t) = s(1, 0, 2) + t(0, 1, 4)$

Basis: $\{(1, 0, 2), (0, 1, 4)\}$

b.) The row space of the matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{(1, 2, -3), (0, 1, -2)\}$

9.) Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ be a collection of vectors.
a.) State what it means for these vectors to be linearly independent.

b.) State what it means for these vectors to span a vector space V.

10.) Given matrices **A** and **B**. Find a matrix **E** such that $\mathbf{EA} = \mathbf{B}$

a.) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

b.) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 4 & -7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 1 & 4 & -7 \end{bmatrix}$$